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A Mathematical Formulation of the Bardeen-Cooper-Schrieffer Theory of Superconductivity

Master's thesis in mathematics

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Abstract

We introduce the model of BCS theory for superconductivity and discuss the assumptions made. We show that minimisers of the BCS functional exist and satisfy the BCS gap equation. Next, we show that, under suitably condition, the critical temperature is exponentially small in both the coupling and the density. Further, we show that the associated energy gap is exponentially small in both the coupling and the density, and that the ratio of the critical temperature and the energy gap tends to the same universal constant in both the weak coupling and the low density limits. Subsequently, we show that the assumptions made in setting up the model are very good for short-range potentials. Finally, we investigate the system with a more general interaction term. Here we show that minimisers exist and satisfy the BCS gap equation.

Contents

1 Introduction

Superconductivity is a physical phenomenon observed in certain materials for temperatures below some critical value T_c. It was first seen experimentally by K. Onnes in 1911, for mercury at very cold temperatures, see [24] for the history of this discovery. Such a superconducting material has many interesting properties. Two of the main ones are that the material can conduct electricity with zero resistivity and that it exhibits a Meissner effect of being a perfect diamagnet. Thus, it repels any magnetic field, and the field-lines go around the superconductor. These effects are not unlimited, meaning that if the current or magnetic field are sufficiently strong they will break the superconductor.

For the theoretical explanation for superconductivity there are two main theories, the Ginzburg-Landau theory and the BCS theory. The Ginzburg-Landau theory for superconductivity is a phenomenological theory developed in 1950 where superconductivity is described in terms of some macroscopic wavefunction ψ minimising a certain functional called the Ginzburg-Landau functional. For temperatures $T > T_c$ above the critical temperature, $\psi \equiv 0$ is zero, but for temperatures $T < T_c$ below the critical temperature $\psi \not\equiv 0$ is non-zero. This means that there is some long-range coherence of the system. The physical interpretation of ψ is however not given in Ginzburg-Landau theory. This came with BCS theory, where it is a sort of wavefunction of Cooper-pairs, see for instance [3, Chapter 5].

In this thesis we will not go much into detail about Ginzburg-Landau theory. We will instead focus on the BCS theory of superconductivity and briefly discuss the link between the two.

The BCS theory of superconductivity was introduced by Bardeen, Cooper and Schrieffer in 1957 in [4] as a microscopic theory of superconductivity. The central part of BCS theory is that superconductivity is caused by a condensation of Cooper pairs. A Cooper pair is a pair of electrons with opposite momenta and spins. Such a pair of electrons behave like a bosonic particle, and so such a condensation is possible. These Cooper-pairs are the charge-carriers. They carry the current in the superconductor.

The BCS theory is also useful in describing superfluidity, see [21]. Part of the more recent interest in BCS theory stems from this additional usage of the theory. For neutral atoms, one should really talk about a superfluid state and not a superconducting state, since the atoms are neutral, so they do not conduct. The important setting here is that of cold fermionic gases, see for instance the references in [9].

We will not go into much details in setting up BCS theory for superfluidity in such Fermionic gases. We will describe the setup of BCS theory for superconductivity. In this case the particles of interest are electrons. Thereafter, one may equally well state all the theorems in the thesis for superfluidity instead of for superconductivity. We will not do this explicitly for the sake of coherence, but one really should change this for the setting of cold neutral fermionic atoms.

Some of the main parts of setting up the model of BCS is to consider what the interaction between the electrons looks like. Firstly there is of course the repulsive (screened) Coulomb interaction. However, in a crystal we also have the phonons, i.e. lattice vibrations. These can interact with the electrons and give rise to an effective electron-electron interaction, when we “integrate out the phonons”. This will be some complicated interaction, which we do not know the looks of. We will model it by some potential V. For the most part, namely in sections 2 to 6 we will assume this to be a multiplication operator in the sense described in those sections.

For the material to be superconducting we need the effective potential V to be sufficiently attractive. This mean that the effective phonon interaction is sufficiently strong in comparison to the (screened) Coulomb interaction. In the physics literature one can find the following explanation that renders this fact probable, see [2, p. 266]. The timescale for the electrons movement past an ion in the crystal lattice is much shorter than the relaxation time for the displacement of the ion. Thus an electron moving past the ion can distort the ion and displace it for a timescale of the relaxation time. Another electron moving past shortly after will thus be affected by the slightly changed ion potential. The ion will not recover to its equilibrium position until after it has been left alone for a timescale of the relaxation time. This gives some intuition for how the phonons lead to an attractive effective electron-electron interaction. This effective interaction is difficult to quantise. This is why we model it by some potential.

One key thing to note here, is that the BCS theory for superconductivity only really describe type I superconductors. The more recently (meaning in the last 30-some years) discovered high temperature (type II) superconductors appear not to be well described by BCS theory. High temperature superconductivity is not fully understood. See for instance [2, 3] for more about this and the physics of BCS and Ginzburg-Landau theory.

In this thesis we will not discuss these various properties of superconductors. Our focus will instead mostly be two-fold. Firstly on developing asymptotic formulas for the critical temperature and the energy gap in different limits, and secondly on the validity of some of the assumptions made in setting up the BCS model. We will not follow BCS's original article or the more modern methods discussed in the physics literature. We will instead go forth more mathematically rigorous and follow the setup of the model and presentation made in [6, 9, 13–16]. This variational approach gives a linear criterion for the critical temperature, in the sense that it is characterised by a certain operator having no negative eigenvalues. The benefit of this more rigorous approach is that the conditions for different results to hold are much more clear.

We now describe the structure of the thesis.

Firstly, in section 2 we will setup the model of writing down the pressure functional associated to the grand canonical potential, that our system should minimise. In doing this we discuss the key assumptions needed to do this. The assumptions made are very clearly motivated by BCS theory [4], and we will discuss what the existence of Cooper-pairs means in our formalism. The existence of such will be our definition for a system being superconducting, motivated again by BCS theory.

Then, we will in section 3 show that minimisers exist for our functional, and that they satisfy an Euler-Lagrange equation, which in this setting is called the BCS gap equation, due to its link to an energy gap. We prove this BCS gap equation with much more detail than the source material [15], where the proof is pretty much only a sketch. This BCS gap equation is an important part of the linear criterion for the critical temperature discussed above. It is this linear criterion, that we will use extensively in section 4 to get formulas for the critical temperature. This section is based on [15, 16].

Thirdly, in section 4 we will consider the critical temperature, below which the system is superconducting. We will develop formulas for this critical temperature in different limits. Firstly, we will consider the limit of a weak coupling, and calculate how the critical temperature behaves in this limit. Here the weak coupling will mean that both the electron-electron interaction and the phonon-interaction are weak. Secondly, we will consider the limit of a low density. This is described as the limit where the chemical potential is small. Our results here agree with what is known in the physics literature. This section is based on [9, 13, 14].

Subsequently, in section 5 we consider the energy gap associated to a system. This energy gap arises as the energy gap in the dispersion relation of certain quasi-particles, in some approximate Hamiltonian. The physics of this is discussed in [2, p. 270-276]. First, we prove that for the limit of weak coupling this energy gap is exponentially small in the coupling. Moreover, it is even exponentially small the same way as the critical temperature is in this limit, and thus we show that the ratio of this energy gap to the critical temperature tends to some universal constant independent of both the potential and the chemical potential. Secondly, we show the new result that in the low density limit the energy gap is exponentially small and that the ratio of the energy gap to the critical temperature tends to the same universal constant as in the weak coupling limit. This section is based on [13] (for the weak coupling limit) and my own work (for the low density limit).

Next, in section 6 we will discuss the validity of some of the assumption made in section 2. In setting up the model the interaction energy will be split in three terms, two of which we neglect in the setup of the model. In this section we will include these terms and do much of the same analysis as we did in section 3. We will see that the assumption of neglecting the two terms is very good for short-range potentials. Including these terms only lead to a sort of renormalisation of the chemical potential. This means that some of the asymptotic results from section 4 still hold for this more correct model where we include these terms. Many of the methods used in this section are the same as in section 4. This section is based on [6].

Then, in section 7 we will consider the model where we no longer assume that the interaction V is given by a multiplication operator. We will discuss what can be done in such a more general case, where V is a more general type of operator. Here we will develop many of the same technical results needed for the proofs of the previous sections to work. This section is based on my own work, partially generalising technical results in [19].

Finally, in section 8 we briefly discuss the model where external fields are included. Here we give an overview of the link between BCS theory and Ginzburg-Landau theory.

2 Setup of the Model

In this section we will describe the model that we consider and all the assumptions we make in setting up the model. We will for simplicity work in units where $\hbar = k_B = 2m = 1$. This section is based on [15].

In quantum mechanics, the state of a particle is given by its wavefunction, which is a normalised element in some Hilbert space. We will denote by $\mathcal{H}$ the Hilbert space of such allowed one-particles states. For a spin-$\frac{1}{2}$ Fermion (for instance an electron) confined to a box $\Lambda \subset \mathbb{R}^3$ we would have $\mathcal{H} = L^2(\Lambda, \mathbb{C}^2)$, where the $\mathbb{C}^2$-part is the spin.

We are interested in Fermions (for instance electrons), so, for a many-particle system, we need the wavefunction to be anti-symmetric in exchanging the particles, this means that the state of a system is given by some element in the Fock-space $\mathcal{F}_{\mathcal{H}} := \bigoplus_{n=0}^{\infty} \bigwedge^n \mathcal{H}$. The n'th component here is the space of n-particle anti-symmetric wavefunctions, i.e. that of allowed n-particle states. An orthonormal basis for the Fock space is given by the Slater determinants

$$\varphi_1 \wedge \cdots \wedge \varphi_n := \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} \operatorname{sgn} \sigma \, \varphi_{\sigma(1)} \otimes \cdots \otimes \varphi_{\sigma(n)},$$

where the set $\{\varphi_i\}_{i \in \mathbb{N}}$ form an orthonormal basis for $\mathcal{H}$. The normalisation is chosen so that the inner product is the usual one on $L^2(\mathbb{R}^{3n})$. The 0'th component $\bigwedge^0 \mathcal{H} = \mathbb{C}\Omega$ represents the vacuum, i.e. Ω is the vacuum state, without any particles.

In order to write down the energy and other related quantities, we need to introduce the creation and annihilation operators. The creation operators $c^\dagger(\phi) : \mathcal{F}_{\mathcal{H}} \to \mathcal{F}_{\mathcal{H}}$ are given by

$$c^\dagger(\phi)(\psi_1 \wedge \cdots \wedge \psi_n) = \phi \wedge \psi_1 \wedge \cdots \wedge \psi_n, \qquad c^\dagger(\phi)\Omega = \phi.$$

The annihilation operators are their adjoints. They are given by

$$c(\phi)(\psi_1 \wedge \cdots \wedge \psi_n) = \sum_{i=1}^{n} (-1)^{i-1} \langle \phi | \psi_i \rangle \, \psi_1 \wedge \cdots \wedge \psi_{i-1} \wedge \psi_{i+1} \wedge \cdots \wedge \psi_n, \qquad c(\phi)\Omega = 0$$

They satisfy the canonical anti-commutation relations

$$\{c(\phi), c^\dagger(\psi)\} = \langle\phi|\psi\rangle\,, \qquad \{c(\phi), c(\psi)\} = 0.$$

The way we will characterise a state is not through an element of the Fock space, but rather through the use of density-matrices. Given a state $\Psi \in \mathcal{F}_\mathcal{H}$, then the associated density matrix $|\Psi\rangle\langle\Psi|$ is the projection onto the subspace spanned by Ψ. We will refer to this density matrix also as the state. Any convex combination of such states $\sum_i \lambda_i |\Psi_i\rangle\langle\Psi_i|$ will again be a state, now just not a pure state.

In general we have the following defining properties of a state ρ

$$\rho : \mathcal{F}_\mathcal{H} \to \mathcal{F}_\mathcal{H}, \qquad 0 \le \rho \le 1, \qquad \operatorname{Tr}\rho = 1.$$

The expectation of some observable A (i.e. a self-adjoint operator) is then $\langle A\rangle_\rho = \operatorname{Tr} A\rho$.

The energy of system is given by the Hamiltonian

$$\mathbb{H} = \sum_{j,k} T_{j,k} c_j^\dagger c_k + \frac{1}{2}\sum_{i,j,k,l} V_{ijkl} c_i^\dagger c_j^\dagger c_l c_k,$$

where $c_j^\dagger = c^\dagger(\varphi_j)$ and $c_j = c(\varphi_j)$. The one-particle part of the energy T is given by

$$T_{jk} = \left\langle \varphi_j \middle| (-i\nabla + A)^2 + W \middle| \varphi_k \right\rangle,$$

where W is the external potential and A is the vector potential. The second term is the energy from the two-particle interaction. It is given by

$$V_{ijkl} = \left\langle \varphi_i(x) \otimes \varphi_j(y) \middle| V(x-y) \middle| \varphi_k(x) \otimes \varphi_l(y) \right\rangle.$$

The number operator, counting the number of particles in a state, is given by $\mathbb{N} = \sum_j c_j^\dagger c_j$. Note that the Hamiltonian given above preserves the number of particles. We will treat the system grand-canonically, that is, not having a fixed number of particles, but instead letting the number of particles vary. Then the state of our system should minimise the pressure functional

$$\mathcal{F}(\rho) = \operatorname{Tr}[(\mathbb{H} - \mu\mathbb{N})\rho] - TS(\rho),$$

where $\mu \in \mathbb{R}$ is the chemical potential, $T \ge 0$ is the temperature, and $S(\rho) = -\operatorname{Tr}(\rho\log\rho)$ is the von-Neumann entropy.

Finding this minimum is too hard. Instead we will restrict to the set of quasi-free states. These we will not define properly, we refer to [23, chap. 10] for a definition. What is usually done in the physics literature is that one approximates the Hamiltonian with a simpler one, and work with that instead. The minimisers of such an approximate Hamiltonian are quasi-free, see [23, chap. 13].

One key property, that quasi-free states have is the following. (One may take this as the definition, if one so desires.)

Theorem 2.1 (Wick's Theorem, [23, Thm. 10.2])**.** *A quasi-free state ρ on $\mathcal{F}_\mathcal{H}$ satisfies that, for all n,*

$$\left\langle c_1^\# \cdots c_{2n}^\# \right\rangle_\rho = \sum_{\sigma\in S_{2n}'} \operatorname{sgn}\sigma \left\langle c_{\sigma(1)}^\# c_{\sigma(2)}^\# \right\rangle_\rho \cdots \left\langle c_{\sigma(2n-1)}^\# c_{\sigma(2n)}^\# \right\rangle_\rho,$$

$$\left\langle c_1^\# \cdots c_{2n+1}^\# \right\rangle_\rho = 0,$$

where $c_j^\#$ is either c_j or $c_j^\dagger$ and $S_{2n}' \subset S_{2n}$ is the subset of permutations σ, with $\sigma(1) < \sigma(3) < \cdots < \sigma(2n-1)$ and $\sigma(2j-1) < \sigma(2j)$ for all $1 \le j \le n$.

From now on ρ will always denote a quasi-free state. Computing the expected value of the energy we then have

$$\langle \mathbb{H} \rangle_\rho = \sum_{j,k} T_{jk} \left\langle c_j^\dagger c_k \right\rangle_\rho + \frac{1}{2} \sum_{i,j,k,l} V_{ijkl} \left(\left\langle c_i^\dagger c_j^\dagger \right\rangle_\rho \langle c_l c_k \rangle_\rho - \left\langle c_i^\dagger c_l \right\rangle_\rho \left\langle c_j^\dagger c_k \right\rangle_\rho + \left\langle c_i^\dagger c_k \right\rangle_\rho \left\langle c_j^\dagger c_l \right\rangle_\rho \right).$$

These 3 terms in the interaction term are referred to as the pairing, the exchange, and the direct term respectively.

Since all expectations can be expressed in terms of two-particle expectations, one would expect that we can somehow encode the information of ρ in a "two-particle state" $\Gamma : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$. This is indeed true. Given ρ we define Γ by

$$\langle \phi_1, \phi_2 | \Gamma | \psi_1, \psi_2 \rangle = \left\langle [c^\dagger(\psi_1) + c(\overline{\psi_2})][c(\phi_1) + c^\dagger(\overline{\phi_2})] \right\rangle_\rho$$

Such a Γ is called a generalised one-particle density matrix. Note that Γ can be written in terms of $\gamma, \alpha : \mathcal{H} \to \mathcal{H}$ given by

$$\langle \phi | \gamma \psi \rangle = \left\langle c^\dagger(\psi) c(\phi) \right\rangle_\rho, \qquad \left\langle \phi \middle| \alpha \overline{\psi} \right\rangle = \langle c(\psi) c(\phi) \rangle_\rho.$$

Then

$$\Gamma = \begin{pmatrix} \gamma & \alpha \\ \alpha^\dagger & 1 - \overline{\gamma} \end{pmatrix}.$$

Here $\overline{\gamma}$ is defined by $\overline{\gamma}\psi = \overline{\gamma \overline{\psi}}$.

For the quasi-free pure states, meaning that they have rank 1, we can classify the operators Γ.

Proposition 2.2 ([23, Thm. 10.4])**.** *There is a bijective correspondence between the set of quasi-free pure states ρ and the set of selfadjoint Γ of the form*

$$\Gamma = \Gamma^2, \qquad \Gamma = \begin{pmatrix} \gamma & \alpha \\ \alpha^\dagger & 1 - \overline{\gamma} \end{pmatrix}, \qquad \gamma \text{ trace-class}$$

satisfying $0 \leq \Gamma \leq 1$.

Remark 2.3. That Γ uniquely identifies ρ is easy to see, this even holds for any quasi-free state.

All this motivates why we may just consider the generalised one-particle density matrices and forget about the associated state ρ.

We can rewrite $\langle \mathbb{H} \rangle_\rho$ in terms of Γ defined above. We want to rewrite this as something simpler. In order to do this, we restrict to the case of spin-$\frac{1}{2}$ particles (for instance electrons) in a box, i.e. $\mathcal{H} = L^2(\Lambda, \mathbb{C}^2)$ for a box $\Lambda = [0, L]^3$. Then an orthonormal basis is given by $\varphi_{k,\sigma} = L^{-3/2} e^{ikx} |\sigma\rangle$, where $k \in \frac{2\pi}{L}\mathbb{Z}^3$ and $\sigma \in \{\uparrow, \downarrow\}$ is the spin. For the sake of simplifying notation we write $|k, \sigma\rangle$ for this state. Introduce the kernels $\gamma_{\sigma,\tau}$ and $\alpha_{\sigma,\tau}$ by

$$\left\langle c_{l,\tau}^\dagger c_{k,\sigma} \right\rangle_\rho = \frac{1}{|\Lambda|} \iint_{\Lambda \times \Lambda} e^{-ikx} \gamma_{\sigma,\tau}(x, y) e^{ily} \, \mathrm{d}y \, \mathrm{d}x,$$
$$\left\langle c_{l,\tau} c_{k,\sigma} \right\rangle_\rho = \frac{1}{|\Lambda|} \iint_{\Lambda \times \Lambda} e^{-ikx} \alpha_{\sigma,\tau}(x, y) e^{-ily} \, \mathrm{d}y \, \mathrm{d}x$$

for $k, l \in \frac{2\pi}{L}\mathbb{Z}^3$ and $\sigma, \tau \in \{\uparrow, \downarrow\}$. Note that exchanging $k \leftrightarrow l$, $\tau \leftrightarrow \sigma$ and $x \leftrightarrow y$ we get that $\alpha_{\sigma,\tau}(x, y) = -\alpha_{\tau,\sigma}(y, x)$. Similarly $\gamma_{\sigma,\tau}(x, y) = \overline{\gamma_{\tau,\sigma}(y, x)}$. Also

$$\left\langle c_{l,\tau}^\dagger c_{k,\sigma}^\dagger \right\rangle_\rho = \overline{\left\langle c_{k,\sigma} c_{l,\tau} \right\rangle_\rho} = \frac{1}{|\Lambda|} \iint_{\Lambda \times \Lambda} e^{ilx} \overline{\alpha_{\tau,\sigma}(x, y)} e^{iky} \, \mathrm{d}y \, \mathrm{d}x.$$

Thus thinking of the kernel $\alpha_{\sigma,\tau}$ as a two-particle wavefunction we have

$$\left\langle c_{l,\tau} c_{k,\sigma} \right\rangle_\rho = \left\langle k, l \middle| \alpha_{\sigma,\tau} \right\rangle \qquad \left\langle c_{l,\tau}^\dagger c_{k,\sigma}^\dagger \right\rangle_\rho = \left\langle \alpha_{\tau,\sigma} \middle| l, k \right\rangle$$

where these inner products are taken in the reduced space, $L^2(\Lambda)^{\otimes 2}$, where we forget about the spin. We will in the following often do this. It will be clear from the context when.

Now, the potential is independent from the spin. Thus, the pairing term becomes

$$\begin{aligned} &\frac{1}{2} \sum_{\substack{k,k',l,l' \\ \sigma,\sigma',\tau,\tau'}} \langle (l,\tau),(k,\sigma)|V|(k',\sigma'),(l',\tau')\rangle \left\langle c_{l,\tau}^\dagger c_{k,\sigma}^\dagger \right\rangle_\rho \left\langle c_{l',\tau'} c_{k',\sigma'} \right\rangle_\rho \\ &\quad = \frac{1}{2} \sum_{\substack{k,k',l,l' \\ \sigma,\sigma',\tau,\tau'}} \langle l,k|V|k',l'\rangle\, \delta_{\tau,\sigma'} \delta_{\sigma,\tau'} \left\langle \alpha_{\tau,\sigma} \middle| l,k \right\rangle \left\langle k',l' \middle| \alpha_{\sigma',\tau'} \right\rangle \\ &\quad = \frac{1}{2} \sum_{\sigma,\tau} \iint_{\Lambda\times\Lambda} V(x-y) |\alpha_{\tau,\sigma}(x,y)|^2 \,\mathrm{d}y\,\mathrm{d}x. \end{aligned}$$

For the direct and exchange term we similarly get

$$\frac{1}{2} \sum_{\sigma,\tau} \iint_{\Lambda\times\Lambda} V(x-y)\gamma_{\tau,\tau}(x,x)\gamma_{\sigma,\sigma}(y,y)\,\mathrm{d}x\,\mathrm{d}y \quad \text{and} \quad -\frac{1}{2} \sum_{\sigma,\tau} \iint_{\Lambda\times\Lambda} |\gamma_{\sigma,\tau}(x,y)|^2 V(x-y)\,\mathrm{d}x\,\mathrm{d}y$$

respectively, using that $\gamma_{\sigma,\tau}(x,y) = \overline{\gamma_{\tau,\sigma}(y,x)}$. In total we thus have

$$\begin{aligned} \langle \mathbb{H} \rangle_\rho \;=\; & \operatorname{Tr}\left[((-i\nabla + A)^2 + W)\gamma \right] \\ & + \frac{1}{2} \sum_{\sigma,\tau} \iint_{\Lambda\times\Lambda} |\alpha_{\sigma,\tau}(x,y)|^2 V(x-y)\,\mathrm{d}x\,\mathrm{d}y \\ & - \frac{1}{2} \sum_{\sigma,\tau} \iint_{\Lambda\times\Lambda} |\gamma_{\sigma,\tau}(x,y)|^2 V(x-y)\,\mathrm{d}x\,\mathrm{d}y \\ & + \frac{1}{2} \sum_{\sigma,\tau} \iint_{\Lambda\times\Lambda} \gamma_{\sigma,\sigma}(x,x)\gamma_{\tau,\tau}(y,y) V(x-y)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

We now restrict to $SU(2)$-invariant states. To define this, let $S \in SU(2)$. Then as explained in [15, Appendix 2] S defines a Bogoliubov transformation W_S with

$$W_S c^\dagger(\psi) W_S^\dagger = c^\dagger(S\psi), \qquad W_S c(\psi) W_S^\dagger = c(S\psi),$$

where S acts pointwise on the spin-part. A state ρ is then $SU(2)$-invariant if

$$\left\langle W_S A W_S^\dagger \right\rangle_\rho = \langle A \rangle_\rho$$

for any operator A. This implies that

$$S^\dagger \gamma S = \gamma, \qquad S^\dagger \alpha \overline{S} = \alpha$$

for all $S \in SU(2)$. If one desires, one may just take this as the definition instead.

That a state is $SU(2)$-invariant is physically that there is no preferred direction for the spin. This is a reasonable assumption for the case where we have no external fields. For the case where we do have a preferred direction, say if there is an external magnetic field, where we have the Zeeman splitting [5, sect. 1.3] we will still have this assumption, i.e. we assume that the energy difference in the Zeeman splitting is very small. This setting is discussed in section 8.

One may then check, that $SU(2)$-invariance implies that

$$\gamma_{\nu,\tau}(x,y) = \gamma(x,y)\delta_{\nu,\tau}, \qquad \alpha_{\nu,\tau}(x,y) = \alpha(x,y)\sigma^y_{\nu,\tau}$$

where $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is the second Pauli-matrix and where α and γ are now kernels of operators on the reduced space $L^2(\Lambda)$. Note that α is symmetric, meaning $\alpha(x,y) = \alpha(y,x)$, and that γ is self-adjoint, meaning $\gamma(x,y) = \overline{\gamma(y,x)}$. We again combine these to a matrix (of operators)

$$\Gamma = \begin{pmatrix} \gamma & \alpha \\ \overline{\alpha} & 1-\overline{\gamma} \end{pmatrix}.$$

Note that by the symmetry of α we have $\overline{\alpha} = \alpha^\dagger$. Plugging this into the expression for the energy above we get

$$\begin{aligned} \langle \mathbb{H} - \mu \mathbb{N} \rangle_\rho - TS(\rho) &= 2\operatorname{Tr}[((-i\nabla + A)^2 - \mu + W)\gamma] - 2TS(\Gamma) \\ &\quad + \iint_{\Lambda\times\Lambda} |\alpha(x,y)|^2 V(x-y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad - \iint_{\Lambda\times\Lambda} |\gamma(x,y)|^2 V(x-y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + 2\iint_{\Lambda\times\Lambda} \gamma(x,x)\gamma(y,y) V(x-y)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Here we have used that $S(\rho) = 2S(\Gamma)$, where the factor of 2 comes from the spin. This is [15, Lem. A.1]. Omitting the direct and exchange term, replacing V by $2V$ and dividing out by 2 we get

$$\mathcal{F}(\Gamma) = \operatorname{Tr}\left[((-i\nabla + A)^2 - \mu + W)\gamma\right] - TS(\Gamma) + \iint_{\Lambda\times\Lambda} |\alpha(x,y)|^2 V(x-y)\,\mathrm{d}x\,\mathrm{d}y. \tag{2.1}$$

Omitting these terms is motivated by the physics of BCS theory. (The Cooper-pairs are the important part.) Below, we discuss how α is related to the Cooper-pairs, and so why the pairing term is the term related to the Cooper-pairs. Mathematically, this simplification is justified by the results in section 6. There we show that for short-range potentials the inclusion of these terms in some sense only lead to a renormalisation of the chemical potential. The details are discussed in section 6.

We now consider the case with absent external fields A and W. In this setting we moreover only consider translation-invariant states, i.e. with

$$\alpha(x,y) = \alpha(x-y) \qquad\qquad \gamma(x,y) = \gamma(x-y).$$

This is motivated by there being no a priori physical reason to prefer some part of configuration space over another. Mathematically it is justified in [15, sect. F], where the model is shown to have its minimum in a translation-invariant state.

Expressing these in terms of their Fourier transforms,

$$\alpha(x-y) = \frac{1}{(2\pi)^{3/2}} \int \hat{\alpha}(p) e^{ip(x-y)}\,\mathrm{d}p, \qquad \gamma(x-y) = \frac{1}{(2\pi)^{3/2}} \int \hat{\gamma}(p) e^{ip(x-y)}\,\mathrm{d}p,$$

and doing a formal infinite volume expansion we get that the energy density is [15]

$$\mathcal{F}(\Gamma) = \int (p^2 - \mu)\gamma(p)\,\mathrm{d}p + \int |\alpha(x)|^2 V(x)\,\mathrm{d}x + T\int \mathrm{Tr}[\Gamma(p)\log\Gamma(p)]\,\mathrm{d}p, \tag{2.2}$$

where we have introduced

$$\Gamma(p) = \begin{pmatrix} \gamma(p) & \hat{\alpha}(p) \\ \overline{\hat{\alpha}(p)} & 1-\gamma(-p) \end{pmatrix}$$

Note that $\hat{\alpha}(-p) = \hat{\alpha}(p)$. This is a slight abuse of notation. This should really have been denoted $\hat{\Gamma}$, and $\hat{\gamma}$ but we will not deal with the original Γ and γ again, so there is no problem. The last term is the entropy $S(\Gamma) = -\int \mathrm{Tr}[\Gamma(p)\log\Gamma(p)]\,\mathrm{d}p$.

Remark 2.4. For the function α here, we can heuristically see it as the expectation of the Cooper-pair with that momentum. The $SU(2)$-invariance dictates that α on the spin-part is off-diagonal, meaning that it only measures pairs of particles with opposite spins. Formally

$$\left\langle c_{p\uparrow}c_{-p\downarrow}\right\rangle_\rho \sim \frac{1}{|\Lambda|}\iint e^{ipx}e^{-ipy}\alpha(x,y)\,\mathrm{d}x\,\mathrm{d}y \sim \hat{\alpha}(p).$$

This motivates why we should call α the Cooper-pair wavefunction. It also motives the following definition.

Definition 2.5. We say that a system is superconducting if, for the minimising $\Gamma = (\gamma,\alpha)$ we have $\alpha \not\equiv 0$.

Not to worry whether such a minimiser exists, we show that in theorem 3.1 below. In BCS-theory, superconductivity is explained as the existence of such Cooper-pairs. Thus, motivated by this, we use the definition of superconductivity stated above.

3 Preliminary Analysis

We now begin our analysis of the model. We show that indeed, minimisers exist and that the critical temperature is described by a certain operator having 0 as its lowest eigenvalue. First, for the existence of minimisers.

3.1 Existence of Minimisers

In this section we show that the BCS functional is bounded from below and attains its minimum. This means that out definition of superconductivity is sensible. This section is based on [15, 16].

Theorem 3.1. *Define the set $\mathcal{D}$ by*

$$\mathcal{D} = \left\{\Gamma(p) = \begin{pmatrix} \gamma(p) & \hat{\alpha}(p) \\ \overline{\hat{\alpha}(p)} & 1-\gamma(-p) \end{pmatrix} : \alpha \in H^1\left(\mathbb{R}^3\right), \gamma \in L^1\left(\mathbb{R}^3, (1+p^2)\,\mathrm{d}p\right), 0 \le \Gamma \le 1\right\}.$$

Let $\mu \in \mathbb{R}$, let $0 \le T < \infty$, and let $V \in L^{3/2}$ be real-valued and reflection-symmetric, meaning that $V(x) = V(-x)$ for every x.

Then $\mathcal{F}$ is bounded from below and attains its minimum on $\mathcal{D}$. Let (γ,α) denote a minimiser. Then α satisfies the BCS gap equation

$$(K_T^\Delta + V)\alpha = 0$$

where K_T^Δ is a multiplication operator in momentum-space and V is a multiplication operator in configuration space. The function K_T^Δ is defined by

$$K_T^\Delta(p) = \frac{E_\Delta(p)}{\tanh\left(\frac{E_\Delta(p)}{2T}\right)}, \qquad E_\Delta(p) = \sqrt{(p^2-\mu)^2 + |\Delta(p)|^2}, \qquad \Delta = 2\widehat{V\alpha} = 2(2\pi)^{-3/2}\hat{V} * \hat{\alpha}.$$

Remark 3.2. Often the BCS gap equation is written in terms of Δ as

$$-\Delta = (2\pi)^{-3/2}\hat{V} * \frac{\Delta}{K_T^\Delta}, \qquad \text{i.e.} \qquad -\Delta(p) = \frac{1}{(2\pi)^{3/2}}\int \hat{V}(p-q)\frac{\Delta(q)}{K_T^\Delta(q)}\,\mathrm{d}q.$$

This is what is used in [15]. We choose our formulation, since it is what we will actually use in the rest of the thesis. That $(K_T^\Delta + V)\alpha = 0$ implies the equation for Δ is clear. The converse implication is discussed in [16].

The reason why the Euler-Lagrange equation for Δ above is called a gap equation is, that Δ in some sense corresponds to an energy gap in BCS theory. This is made precise in [16, Appendix A].

The non-vanishing of α is the same as the non-vanishing of Δ. We show below, that there is some critical temperature T_c, such that for $T > T_c$ we have $\alpha \equiv 0$ and for $T < T_c$ we have $\alpha \not\equiv 0$. Hence we have a phase transition. In this phase transition we have a broken symmetry, that of a global $U(1)$-symmetry. (It is easy to see that we may change the phase of α (hence also of Δ) by a global constant.) The associated Goldstone modes are gapped (this is where Δ corresponds to an energy gap), see for instance [2, p. 270-276]. This further motivates, why our definition of superconductivity is a good one. We will discuss this energy gap more in section 5.

Remark 3.3 (On the assumptions on V)**.** We assume here that V is reflection-symmetric. This assumption is not made in the original paper [16], but is made in [15], which is what this proof is based on. It is an error, that this assumption is not stated in the original paper. We need this assumption in order to get the Euler-Lagrange equations (the BCS gap equation) for the minimiser of the functional. The proof of these Euler-Lagrange equations in [15, Proof of prop. 3.1] is short on details and it is not discussed how the reflection-symmetry of V is needed.

We provide our own proof of the Euler-Lagrange equations here (based somewhat on the ideas in [15]). After proving the Euler-Lagrange equations we discuss the necessity of assuming that V is reflection-symmetric in remark 3.4 below. Here we show that it is almost necessary to make this assumption.

The assumption that V is reflection-symmetric is also quite physical. Often, one might even have a homogeneous material, where V would even be radial.

We now prove this theorem. Suppose first $T > 0$.

3.1.1 Positive Temperature

We bound $\frac{1}{4}\int (p^2-\mu)\gamma(p)\,\mathrm{d}p - TS(\Gamma)$ from below. By writing down the dependence of the eigenvalues of $\Gamma(p)$ on α one sees that this expression is increasing in $|\hat{\alpha}|$, so that for a lower bound we may set $\alpha \equiv 0$. Note that if $(\gamma, \alpha) \in \mathcal{D}$ then also $(\gamma, 0) \in \mathcal{D}$, so this is still a valid state. Now, twice the value of this expression is

$$\begin{aligned}\frac{1}{4}\int (p^2-\mu)(\gamma(p)+\gamma(-p))\,\mathrm{d}p + T\int &\gamma(p)\log\gamma(p) + (1-\gamma(p))\log(1-\gamma(p)) \\ &+ \gamma(-p)\log\gamma(-p) + (1-\gamma(-p))\log(1-\gamma(-p))\,\mathrm{d}p \\ = 2\bigg(\frac{1}{4}\int (p^2-\mu)\gamma(p)\,\mathrm{d}p &+ T\int \gamma(p)\log\gamma(p) + (1-\gamma(p))\log(1-\gamma(p))\,\mathrm{d}p\bigg)\end{aligned}$$

Now, bounding the integrand pointwise in p by varying the values of $\gamma(p)$ we get

$$\frac{1}{4}\int (p^2-\mu)\gamma(p)\,\mathrm{d}p - TS(\Gamma) \geq C_1 := -T\int \log\left(1+e^{-\frac{1}{4T}(p^2-\mu)}\right)\,\mathrm{d}p > -\infty.$$

Now [19, sect. 11.3] gives that p^2+V is bounded from below, i.e. $0 \geq C_2 := \inf\operatorname{spec}(p^2/4+V) > -\infty$. Additionally by $0 \leq \Gamma \leq 1$ we get $|\hat{\alpha}(p)|^2 \leq \gamma(p)$. Hence

$$\frac{1}{4}\int p^2\gamma(p)\,\mathrm{d}p + \int |\alpha(x)|^2 V(x)\,\mathrm{d}x \geq C_2\int |\hat{\alpha}(p)|^2\,\mathrm{d}p \geq C_2\int \gamma(p)\,\mathrm{d}p.$$

Combining this we arrive at

$$\begin{aligned}\mathcal{F}(\Gamma) &\geq C_1 + \int\left(\frac{2p^2}{4}-\frac{3\mu}{4}+C_2\right)\gamma(p)\,\mathrm{d}p\\ &\geq C_1 + \int\left(\frac{p^2}{4}-\frac{3\mu}{4}-\frac{1}{4}+C_2\right)\gamma(p)\,\mathrm{d}p + \frac{1}{8}\int(1+p^2)\gamma(p)\,\mathrm{d}p + \frac{1}{8}\int(1+p^2)|\hat{\alpha}(p)|^2\,\mathrm{d}p\\ &\geq -A + \frac{1}{8}\,\|\gamma\|_{L^1(\mathbb{R}^3,(1+p^2)\,\mathrm{d}p)} + \frac{1}{8}\,\|\alpha\|^2_{H^1(\mathbb{R}^3)},\end{aligned}$$

where $-A := C_1 - \int\left[\frac{p^2}{4}-\frac{3\mu}{4}-\frac{1}{4}+C_2\right]_-\,\mathrm{d}p \leq 0$. Here $[\cdot]_- = -\min\{\cdot,0\}$ denote the negative part. This shows that $\mathcal{F}$ is bounded from below as claimed.

Let now $\Gamma_n = (\gamma_n,\alpha_n)$ be a minimising sequence with $\mathcal{F}(\Gamma_n) \leq 0$. Then $\|\alpha_n\|^2_{H^1} \leq 8A$, so α_n is bounded in H^1, so some subsequence (which we continue to denote by α_n) will converge weakly to some $\alpha \in H^1$ by Banach-Alaoglu. By [19, Thm. 11.4] we get that V is weakly H^1 form-continuous, i.e. $\int |\alpha_n(x)|^2 V(x)\,\mathrm{d}x \to \int |\alpha(x)|^2 V(x)\,\mathrm{d}x$ for this subsequence α_n. Similarly we have that γ_n is uniformly bounded in $L^1((1+p^2)\,\mathrm{d}p)$. Since also $0 \leq \gamma_n(p) \leq 1$ we have that γ_n is uniformly bounded in L^∞. Thus γ_n is uniformly bounded in $L^q((1+p^2)\,\mathrm{d}p)$ for any $1 < q < \infty$. So fix some $1 < q < \infty$.

The remaining part of the functional

$$\mathcal{F}^0(\Gamma) = \int (p^2-\mu)\gamma(p)\,\mathrm{d}p - TS(\Gamma)$$

is convex. Now, with another application of Banach-Alaoglu together with Mazur's lemma [19, Thm. 2.13] (see [17, Thm. 2.9 - Lem. 2.11] for the details) we find a sequence $\tilde{\Gamma}_n = (\tilde{\gamma}_n,\tilde{\alpha}_n)$ of convex combinations of the Γ_n such that $\tilde{\alpha}_n \to \alpha$ pointwise a.e. and in H^1 and $\tilde{\gamma}_n \to \gamma$ pointwise a.e. and in $L^q((1+p^2)\,\mathrm{d}p)$. Moreover, $\liminf \mathcal{F}^0(\tilde{\Gamma}_n) \leq \liminf \mathcal{F}^0(\Gamma_n)$. Also, $\tilde{\alpha}_n$ still converges weakly to α and so $\int |\tilde{\alpha}_n|^2 V\,\mathrm{d}x \to \int |\alpha|^2 V\,\mathrm{d}x$. Clearly $\Gamma \in \mathcal{D}$. It remains to be checked that Γ really is a minimiser.

To see that Γ is a minimiser, we first note that the integrand in the expression for $\mathcal{F}^0(\Gamma_n)$ is bounded from below by the integrable function $-T\log\left(1+e^{-\frac{1}{T}(p^2-\mu)}\right)$. This is again seen by varying $\gamma(p)$ for each fixed p. Hence we get that the same pointwise bound hold for the sequence of convex combinations, and so we may apply Fatou's lemma

$$\mathcal{F}(\Gamma) = \mathcal{F}(\liminf\tilde{\Gamma}_n) \leq \liminf\mathcal{F}(\tilde{\Gamma}_n) = \inf_{\tilde{\Gamma}\in\mathcal{D}}\mathcal{F}(\tilde{\Gamma}).$$

We conclude that a minimiser of the functional $\mathcal{F}$ exist. Now, we show the BCS gap equation for this minimiser $\Gamma = (\gamma,\alpha)$.

First, we claim that $\gamma(p) = \gamma(-p)$. To see this, observe that $\tilde{\Gamma}(p) = \begin{pmatrix}\gamma(-p) & \hat{\alpha}(p)\\ \bar{\hat{\alpha}}(p) & 1-\gamma(p)\end{pmatrix} \in \mathcal{D}$ is an allowed state. Moreover, $\tilde{\Gamma}$ has the eigenvalues $1-\lambda_{1,2}$, where $\lambda_{1,2}$ denotes the eigenvalues for

Γ. (These can just be computed.) Thus $S(\tilde{\Gamma}) = S(\Gamma)$ and so $\mathcal{F}(\tilde{\Gamma}) = \mathcal{F}(\Gamma)$. Also, $\mathcal{F}^0$ defined above is strictly convex. Then we have

$$\begin{aligned}\mathcal{F}(\Gamma) \leq \mathcal{F}\left(\frac{\Gamma+\tilde{\Gamma}}{2}\right) &= \int |\alpha(x)|^2 V(x)\,\mathrm{d}x + \mathcal{F}^0\left(\frac{\Gamma+\tilde{\Gamma}}{2}\right) \\ &\overset{(*)}{\leq} \int |\alpha(x)|^2 V(x)\,\mathrm{d}x + \frac{1}{2}\mathcal{F}^0(\Gamma) + \frac{1}{2}\mathcal{F}^0(\tilde{\Gamma}) = \frac{1}{2}\mathcal{F}(\Gamma) + \frac{1}{2}\mathcal{F}(\tilde{\Gamma}) = \mathcal{F}(\Gamma)\end{aligned}$$

where $(*)$ is an equality if and only if $\tilde{\Gamma} = \Gamma$. We conclude that $\Gamma = \tilde{\Gamma}$, and so that $\gamma(p) = \gamma(-p)$ for every p. Since also $\hat{\alpha}(p) = \hat{\alpha}(-p)$ for every p we see that $\Gamma(p) = \Gamma(-p)$.

Now, since the entropy $-t\log t$ has unbounded derivative near 0 and 1 we have that for the minimising Γ its eigenvalues are never 0 or 1. More precisely, the eigenvalues of Γ are bounded away from 0 and 1 on any given compact set.

Let now $\tilde{\Gamma}$ be any other state such that $\tilde{\Gamma} - \Gamma$ is of compact support. Then for sufficiently small $t > 0$, the matrix $\Gamma + t(\tilde{\Gamma} - \Gamma)$ will also have its eigenvalues bounded away from 0 and 1 on the compact set $\operatorname{supp}(\tilde{\Gamma} - \Gamma)$. Hence by a standard dominated convergence argument we get that the entropy $S(\Gamma + t(\tilde{\Gamma} - \Gamma))$ is differentiable at $t = 0$ and thus

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{F}(\Gamma + t(\tilde{\Gamma} - \Gamma)) \geq 0,$$

where the limit in the derivative is $t \searrow 0$. For any state $\tilde{\Gamma}$ we have that the entropy is given by $S(\tilde{\Gamma}) = -\frac{1}{2}\int \operatorname{Tr}[\tilde{\Gamma}\log\tilde{\Gamma} + (1-\tilde{\Gamma})\log(1-\tilde{\Gamma})]\,\mathrm{d}p$. (To see this, simply compute the eigenvalues of $\tilde{\Gamma}$ and $1 - \tilde{\Gamma}$ at any point.) Hence, we compute that for the $\tilde{\Gamma}$ considered above we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{F}(\Gamma + t(\tilde{\Gamma} - \Gamma)) = \frac{1}{2}\int \operatorname{Tr}\left[(\tilde{\Gamma} - \Gamma)\left(H_\Delta + T\log\frac{\Gamma}{1-\Gamma}\right)\right]\,\mathrm{d}p,$$

with $H_\Delta(p) = \begin{pmatrix} p^2 - \mu & \Delta(p) \\ \overline{\Delta(p)} & -p^2 + \mu \end{pmatrix}$. Additionally, since the eigenvalues of Γ stay away from 0 and 1, the state $\tilde{\Gamma}' := \Gamma - (\tilde{\Gamma} - \Gamma)$ is also allowed (meaning that $\Gamma + t(\tilde{\Gamma}' - \Gamma)$ is a state for small $t > 0$) and so we can compute the same for this state and thus we get the equality

$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{F}(\Gamma + t(\tilde{\Gamma} - \Gamma)) = \frac{1}{2}\int \operatorname{Tr}\left[(\tilde{\Gamma} - \Gamma)\left(H_\Delta + T\log\frac{\Gamma}{1-\Gamma}\right)\right]\,\mathrm{d}p.$$

We now claim that

$$H_\Delta + T\log\frac{\Gamma}{1-\Gamma} = 0.$$

To see this first note that

$$\tilde{\Gamma}(p) - \Gamma(p) = \begin{pmatrix} a(p) & b(p) \\ \overline{b(p)} & -a(-p) \end{pmatrix}$$

for some $a = \tilde{\gamma} - \gamma \in L^1((1+p^2)\,\mathrm{d}p)$ and $b = \hat{\tilde{\alpha}} - \hat{\alpha} \in H^1$. Moreover any matrix function $A(p) = \begin{pmatrix} a(p) & b(p) \\ \overline{b(p)} & -a(-p) \end{pmatrix}$ with $a, b \in C_0^\infty$ and $b(p) = b(-p)$ is of the form $A = \frac{1}{\varepsilon}\left(\tilde{\Gamma} - \Gamma\right)$ for some small $\varepsilon > 0$ and $\tilde{\Gamma} \in \mathcal{D}$. Hence we have that

$$\int \operatorname{Tr}\left[A\left(H_\Delta + T\log\frac{\Gamma}{1-\Gamma}\right)\right]\,\mathrm{d}p = 0$$

for all such A. Let $p_0 \in \mathbb{R}^3$ be fixed and choose

$$a(p) = a\left[\chi\left(N(p_0 - p)\right) + \chi\left(N(p_0 + p)\right)\right] \qquad b(p) = b\left[\chi\left(N(p_0 - p)\right) + \chi\left(N(p_0 + p)\right)\right]$$

for any numbers $a \in \mathbb{R}, b \in \mathbb{C}$ and $N > 0$ and a cut-off function χ of compact support satisfying $\chi(0) = 1$. Letting then $N \to \infty$ we get the pointwise equality

$$\operatorname{Tr}\left[\begin{pmatrix} a & b \\ \overline{b} & -a \end{pmatrix}\left(\left(H_\Delta + T\log\frac{\Gamma}{1-\Gamma}\right)(p_0) + \left(H_\Delta + T\log\frac{\Gamma}{1-\Gamma}\right)(-p_0)\right)\right] = 0.$$

Now, varying a and b we get

$$\left(H_\Delta + T\log\frac{\Gamma}{1-\Gamma}\right)(p_0) + \left(H_\Delta + T\log\frac{\Gamma}{1-\Gamma}\right)(-p_0) = k$$

for some constant k since constants are the only matrices, that are orthogonal to all trace-0 matrices. Above we showed that $\Gamma(p_0) = \Gamma(-p_0)$ and that $\operatorname{Tr}\Gamma(p_0) = 1$. Hence $\Gamma(p_0)$ has the eigenvalues λ and $1-\lambda$ for some $0 < \lambda < 1$. Additionally $\operatorname{Tr} H_\Delta = 0$ and so

$$2k = \operatorname{Tr} k = \operatorname{Tr}\left(H_\Delta(p_0) + H_\Delta(-p_0) + 2T\log\frac{\Gamma}{1-\Gamma}(p_0)\right) = 2T\log\frac{\lambda}{1-\lambda} + 2T\log\frac{1-\lambda}{1-(1-\lambda)} = 0.$$

That is,

$$\left(H_\Delta + T\log\frac{\Gamma}{1-\Gamma}\right)(p_0) + \left(H_\Delta + T\log\frac{\Gamma}{1-\Gamma}\right)(-p_0) = 0. \tag{3.1}$$

In order to show that both summands are zero we need the reflection symmetry of V. Since V and α are reflection-symmetric, so is $\Delta = 2\widehat{V\alpha}$. Thus $H_\Delta(p) = H_\Delta(-p)$ for every p. Thus the two summands above are the same. Since they sum to zero and p_0 was arbitrary we thus get

$$H_\Delta + T\log\frac{\Gamma}{1-\Gamma} = 0.$$

This is the Euler-Lagrange equation for our functional.

Note that $H_\Delta^2 = E_\Delta^2$. Thus we have, by solving the equation above for Γ that

$$\Gamma(p) = \frac{1}{1 + e^{\frac{1}{T}H_\Delta(p)}} = \frac{1}{2} - \frac{1}{2}\tanh\frac{H_\Delta(p)}{2T} = \frac{1}{2} - \frac{1}{2}\frac{H_\Delta(p)}{E_\Delta(p)}\tanh\frac{E_\Delta(p)}{2T} = \begin{pmatrix} \frac{1}{2} - \frac{p^2-\mu}{2K_T^\Delta(p)} & \frac{-\Delta(p)}{2K_T^\Delta(p)} \\ \frac{-\overline{\Delta(p)}}{2K_T^\Delta(p)} & \frac{1}{2} + \frac{p^2-\mu}{2K_T^\Delta(p)} \end{pmatrix},$$

since $\tanh$ is an odd function, so $\tanh H_\Delta = \frac{H_\Delta}{E_\Delta}\tanh E_\Delta$. Hence we have the Euler-Lagrange equations

$$\begin{pmatrix} \gamma(p) & \hat{\alpha}(p) \\ \overline{\hat{\alpha}(p)} & 1-\gamma(-p) \end{pmatrix} = \Gamma(p) = \begin{pmatrix} \frac{1}{2} - \frac{p^2-\mu}{2K_T^\Delta(p)} & \frac{-\Delta(p)}{2K_T^\Delta(p)} \\ \frac{-\overline{\Delta(p)}}{2K_T^\Delta(p)} & \frac{1}{2} + \frac{p^2-\mu}{2K_T^\Delta(p)} \end{pmatrix}. \tag{3.2}$$

In particular $\hat{\alpha} = \frac{-\Delta}{2K_T^\Delta}$. Recalling the definition of Δ we thus have the BCS gap equation. Note that convolving with $\hat{V}$ gives the BCS gap equation for Δ. This finishes the proof in the case $T > 0$.

Remark 3.4 (Necessity of assuming that V is reflection-symmetric)**.** We can prove that we indeed need V to be (almost) reflection-symmetric. The two summands in equation (3.1) above only potentially differ in the terms $\Delta(p_0)$ and $\Delta(-p_0)$. Since we need both to be zero we get that these must be the same, i.e. that Δ must be reflection-symmetric. Then by Fourier transforming we get that the product $V\alpha$ must be reflection-symmetric. Since α already is reflection-symmetric we thus get that $V(x) = V(-x)$ for every x where $\alpha(x) \neq 0$. Of course this might not mean for every $x \in \mathbb{R}^3$, and this is what we mean by V being almost reflection-symmetric.

Since α depends on V in some complicated way, we assumed that V was reflection-symmetric everywhere. This is a much more natural assumption than assuming that V is only reflection-symmetric on the set where α does not vanish. (Verifying such an assumption would also be infeasible.)

Now we deal with the case $T = 0$.

3.1.2 Zero Temperature

For the existence of minimiser this case is analogous to the $T > 0$ case, only with

$$C_1 = \inf_{\Gamma\in\mathcal{D}} \frac{1}{4}\int (p^2-\mu)\gamma(p)\,\mathrm{d}p = \frac{1}{4}\int (p^2-\mu)\mathbb{1}_{\{p^2<\mu\}}\,\mathrm{d}p = \begin{cases} \frac{-2\pi\mu^{5/2}}{15} & \text{if } \mu > 0, \\ 0 & \text{if } \mu \leq 0 \end{cases} > -\infty$$

instead. We conclude that minimiser do exists.

In order to show the BCS gap equation we show that $\Gamma = \mathbb{1}_{\{H_\Delta<0\}}$, the projection onto the negative eigenspace for H_Δ. Then pointwise in p

$$\Gamma(p) = \mathbb{1}_{\{H_\Delta<0\}}(p) = \lim_{T\to 0}\frac{1}{1+e^{\frac{1}{T}H_\Delta(p)}} = \lim_{T\to 0}\begin{pmatrix} \frac{1}{2} - \frac{p^2-\mu}{2K_T^\Delta(p)} & \frac{-\Delta(p)}{2K_T^\Delta(p)} \\ \frac{-\Delta(p)}{2K_T^\Delta(p)} & \frac{1}{2} + \frac{p^2-\mu}{2K_T^\Delta} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - \frac{p^2-\mu}{2K_0^\Delta(p)} & \frac{-\Delta(p)}{2K_0^\Delta(p)} \\ \frac{-\Delta(p)}{2K_0^\Delta(p)} & \frac{1}{2} + \frac{p^2-\mu}{2K_0^\Delta(p)} \end{pmatrix}$$

and we may conclude the gap equation as we did in the $T > 0$ case.

First we show, that Γ is indeed a projection. The conditions $0 \leq \Gamma \leq 1$ give that

$$0 \leq \det\Gamma(p) = \gamma(p)(1-\gamma(-p)) - |\hat{\alpha}(p)|^2, \qquad 0 \leq \det(1-\Gamma(p)) = \gamma(-p)(1-\gamma(p)) - |\hat{\alpha}(p)|^2.$$

For every fixed p and fixed value of $\hat{\alpha}(p)$, these inequalities define some strictly convex set of allowed pairs $(\gamma(p), \gamma(-p))$ in the unit square. This set is reflection-symmetric in the line $\gamma(p) = \gamma(-p)$. For this fixed value of $\hat{\alpha}(p)$ we should have $\gamma(p) + \gamma(-p)$ maximal for $p^2 \leq \mu$ and minimal for $p^2 > \mu$, since Γ is a minimiser. Hence this occurs in one of the two points where we have equalities

$$\gamma(-p)(1-\gamma(p)) = \gamma(p)(1-\gamma(-p)) = |\hat{\alpha}(p)|^2.$$

Thus both $\Gamma(p)$ and $\mathbb{1} - \Gamma(p)$ have determinant 0, and so Γ is a rank one projection. Moreover, $\gamma(p) = \gamma(-p)$. (In order to see this argument more clearly, it can prove helpful to make a sketch of the allowed set of pairs $(\gamma(p), \gamma(-p))$ given by the inequalities above.)

We may compute the derivative exactly as for the $T > 0$ case, only now we don't have to deal with the trouble of whether the entropy is differentiable. That is, for any state $\tilde{\Gamma} \in \mathcal{D}$ we have

$$0 \leq \int \operatorname{Tr}\left[(\tilde{\Gamma} - \Gamma)H_\Delta\right]\,\mathrm{d}p.$$

Let p_0 be fixed and consider the state

$$\tilde{\Gamma}(p) = \left(1 - \frac{\chi(N(p_0-p)) + \chi(N(p_0+p))}{2}\right)\Gamma(p) + \frac{\chi(N(p_0-p)) + \chi(N(p_0+p))}{2}\mathbb{1}_{\{H_\Delta<0\}}(p)$$

with $0 \leq \chi \leq 1$ a smooth cut-off function supported in $B(0,1)$ with $\chi(0) = 1$. Note that this is an allowed state since $H_\Delta(p) = H_\Delta(-p)$ by the reflection-symmetry of V. Letting $N \to \infty$ we get the pointwise inequality

$$0 \leq \operatorname{Tr}\left[\frac{1}{2}\left(-\Gamma(p_0) + \mathbb{1}_{\{H_\Delta<0\}}(p_0)\right)H_\Delta(p_0)\right] + \operatorname{Tr}\left[\frac{1}{2}\left(-\Gamma(-p_0) + \mathbb{1}_{\{H_\Delta<0\}}(-p_0)\right)H_\Delta(-p_0)\right].$$

Now, $\Gamma(-p) = \Gamma(p)$ and $H_\Delta(-p) = H_\Delta(p)$ for any p and p_0 was arbitrary. Thus

$$0 \le \operatorname{Tr}\left[(-\Gamma + \mathbb{1}_{\{H_\Delta<0\}})H_\Delta\right].$$

Since H_Δ is hermitian with eigenvalues $\pm E_\Delta$ we have that

$$H_\Delta = E_\Delta\left(\mathbb{1}_{\{H_\Delta>0\}} - \mathbb{1}_{\{H_\Delta<0\}}\right) = E_\Delta\left(2\mathbb{1}_{\{H_\Delta>0\}} - 1\right).$$

We conclude that $\operatorname{Tr}\Gamma\mathbb{1}_{\{H_\Delta>0\}} \le 0$. On the other hand, $\operatorname{Tr}\Gamma\mathbb{1}_{\{H_\Delta>0\}} \ge 0$ is clearly positive as both are rank one projections. We conclude that Γ and $\mathbb{1}_{\{H_\Delta>0\}}$ are orthogonal and thus that $\Gamma = \mathbb{1}_{\{H_\Delta<0\}}$ as desired. This concludes the proof in the $T = 0$ case.

Remark 3.5. Note that we proved above that for any $T \ge 0$ the minimiser Γ is reflection-symmetric, i.e. it satisfies that $\Gamma(p) = \Gamma(-p)$.

Remark 3.6. Above we only concluded the Euler-Lagrange equation for α. There is of course also one for γ. Going back to equation (3.2) we see that the Euler-Lagrange equation for γ is

$$\gamma(p) = \frac{1}{2} - \frac{p^2-\mu}{2K_T^\Delta(p)},$$

which of course also holds in the $T = 0$ case.

3.2 Linear Criterion for the Critical Temperature

We use the BCS gap equation to relate the existence of a superconducting phase to properties of the operator $K_T^\Delta + V$. This section is based on [15].

Define the normal state $\Gamma_0 = \begin{pmatrix}\gamma_0 & 0\\ 0 & 1-\gamma_0\end{pmatrix}$ with $\gamma_0(p) = \frac{1}{1+e^{\frac{1}{T}(p^2-\mu)}}$. This state is seen to minimise the non-interacting, $V = 0$, case of the BCS functional. This is again a simple minimisation problem varying $\gamma(p)$ pointwise for fixed p. We have the following theorem

Theorem 3.7. *Let $V \in L^{3/2}$ be real-valued and reflection-symmetric, $\mu \in \mathbb{R}$ and $0 \le T < \infty$. Then the following are equivalent*

(i) The normal state Γ_0 is not a minimiser, i.e. $\inf_{\Gamma\in\mathcal{D}}\mathcal{F}(\Gamma) < \mathcal{F}(\Gamma_0)$,

(ii) There exists $\Gamma = (\gamma, \alpha)$ with $\alpha \ne 0$ non-vanishing satisfying the BCS gap equation $(K_T^\Delta + V)\alpha = 0$,

(iii) The operator $K_T^0 + V$ has at least one negative eigenvalue.

This leads us to the following definition of the critical temperature

Definition 3.8. The *critical temperature* of a system is $T_c = T_c(V) := \inf\{T \ge 0 : K_T^0 + V \ge 0\}$.

The above theorem says that for $T < T_c$ the system is superconducting. Since K_T^0 is monotone in T we get that for $T > T_c$ the system is not superconducting. We now prove the theorem.

Proof. The claim $(i) \implies (ii)$ is proven already. For the implication $(ii) \implies (iii)$ we do the following. First, the reader is invited to verify herself, that for any fixed p the function $[0,\infty) \ni |\Delta| \mapsto K_T^\Delta(p)$ is strictly increasing. Thus for our function Δ we have that $K_T^\Delta(p) > K_T^0(p)$ for all p, where Δ and therefore also $\hat{\alpha}$ is non-vanishing. Additionally we have that $K_T^0 + V \le K_T^\Delta + V$ as operators. We thus conclude that

$$\langle\alpha|K_T^0 + V|\alpha\rangle < \langle\alpha|K_T^\Delta + V|\alpha\rangle = 0.$$

We claim that this implies that $K_T^0 + V$ has a negative eigenvalue. This follows once we prove that the essential spectrum of $K_T^0 + V$ starts at $2T \geq 0$, meaning that it is contained in $[2T, \infty)$.

Define the function $k(p) = 2T\left(\frac{\frac{p^2-\mu}{2T}}{\tanh\frac{p^2-\mu}{2T}} - 1\right)$. Assume first that $\mu \geq 0$. Then for $|p| \to \infty$ we have $k(p) \sim p^2$. Moreover, since $|p| = \mu$ is a minimum with $k(p) = 0$ we thus have that $k(p) \geq c(p^2 - 2\mu)$ for some constant $c > 0$, which we can choose arbitrarily small, and all p . That the essential spectrum of $K_T^0 + V$ starts at $2T$ is exactly that the essential spectrum of $k + V$ starts at 0. The spectral values of $k + V$ are higher than the spectral values of $c(p^2 - 2\mu) + V$ by the inequality $k + V \geq c(p^2 - 2\mu) + V$. Moreover, the essential spectrum of $c(p^2 - 2\mu) + V = c\left(p^2 + \frac{V}{c}\right) - 2c\mu$ starts at $-2c\mu$ by [19, Thm. 11.6]. (Indeed [19, Thm. 11.6] says that any negative spectral value of $p^2 + V$ is an eigenvalue. Alternatively the CLR-bound [20, Thm. 4.1] gives that the number of negative spectral values of $p^2 + V$ is even finite.)

Taking the limit $c \searrow 0$ we get that the essential spectrum for $k + V$ starts at 0. It follows that the essential spectrum of $K_T^0 + V$ starts at $2T$ as stated and thus that $K_T^0 + V$ has a negative eigenvalue. For $\mu < 0$ the same applies, now only $k(p) \geq cp^2$ instead.

Now we prove that $(iii) \implies (i)$. First we assume $T > 0$. Let $\varphi \in C_0^\infty$ be some (reflection-symmetric) function (in Fourier space) and consider for small t (both positive and negative) the function

$$t \mapsto \mathcal{F}\left(\Gamma_0 + t\begin{pmatrix} 0 & \varphi \\ \bar{\varphi} & 0 \end{pmatrix}\right),$$

where t small means that the argument in $\mathcal{F}$ lies in $\mathcal{D}$. If Γ_0 is not a stationary point, then (i) is clearly true, so we may assume that Γ_0 is a stationary point. We thus intend to compute the second derivative of this function at $t = 0$. The interaction part of the functional is not problematic and gives for the second derivative $2\langle\varphi|V|\varphi\rangle = 2\int V|\check{\varphi}|^2 \, \mathrm{d}x$. The kinetic energy term is constant. We thus turn our attention to the entropy.

Define $G := \begin{pmatrix} 0 & \varphi \\ \bar{\varphi} & 0 \end{pmatrix}$ and $f(s) = \frac{1}{2}(s\log s + (1-s)\log(1-s))$. We thus have that the entropy is given by $S(\Gamma_0 + tG) = -\int \operatorname{Tr} f(\Gamma_0 + tG) \, \mathrm{d}p$. For every fixed p we have (suppressing in the notation that all functions are evaluated at p)

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Tr} f(\Gamma_0 + tG) = \operatorname{Tr}[f'(\Gamma_0 + tG)G] = \frac{1}{2\pi i}\oint_C f'(z)\operatorname{Tr}\left[\frac{1}{z - \Gamma_0 - tG}G\right] \mathrm{d}z,$$

where $C = \partial B\left(\frac{1}{2}, \frac{1}{2} - \varepsilon\right)$ is large enough to encircle the eigenvalues of $\Gamma_0(p) + tG(p)$. (In order to see this version of the Cauchy integral formula, one can just expand the integrand as a power series in $\frac{1}{z}$, exchange integration and summation and use the ordinary Cauchy integral formula for derivatives, see [7].) Now, taking the derivative of this we have

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} \operatorname{Tr} f(\Gamma_0 + tG) &= \frac{1}{2\pi i}\oint_C f'(z)\operatorname{Tr}\left[\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\frac{1}{z - \Gamma_0 - tG}G\right] \mathrm{d}z \\ &= \frac{1}{2\pi i}\oint_C f'(z)\operatorname{Tr}\left[\frac{1}{z - \Gamma_0}G\frac{1}{z - \Gamma_0}G\right] \mathrm{d}z, \end{aligned}$$

since for a matrix function $U(t)$ we have $\frac{\mathrm{d}}{\mathrm{d}t}\left(U^{-1}\right) = -U^{-1}\frac{\mathrm{d}}{\mathrm{d}t}UU^{-1}$. Now, one may compute that $\{\Gamma_0, G\} = G$ and so

$$\left\{G, \frac{1}{z - \Gamma_0}\right\} = \frac{1}{z - \Gamma_0}\{G, z - \Gamma_0\}\frac{1}{z - \Gamma_0} = (2z - 1)\frac{1}{z - \Gamma_0}G\frac{1}{z - \Gamma_0}.$$

Using this we get

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} \operatorname{Tr} f(\Gamma_0 + tG) &= \frac{1}{2\pi i}\oint_C f'(z) \operatorname{Tr}\left[\frac{1}{z-\Gamma_0}G\frac{1}{z-\Gamma_0}G\right] \mathrm{d}z \\
&= \frac{1}{2\pi i}\oint_C \frac{f'(z)}{2z-1} \operatorname{Tr}\left[\left\{\frac{1}{z-\Gamma_0}, G\right\} G\right] \mathrm{d}z \\
&= \frac{1}{2\pi i}\oint_C \frac{2f'(z)}{2z-1} \operatorname{Tr}\left[\frac{1}{z-\Gamma_0}G^2\right] \mathrm{d}z \\
&= \operatorname{Tr}\left[\frac{2f'(\Gamma_0)}{2\Gamma_0 - 1}G^2\right]
\end{aligned}$$

since $f'(s) = \frac{1}{2}\log\left(\frac{s}{1-s}\right)$ so $f'\left(\frac{1}{2}\right) = 0$. Now, $\Gamma_0 = \frac{1}{1+e^{\frac{1}{T}H_0}}$ so one computes $\frac{\Gamma_0}{1-\Gamma_0} = e^{-\frac{1}{T}H_0}$ and $2\Gamma_0 - 1 = -\tanh\left(\frac{H_0}{2T}\right)$, where $H_0(p) = \begin{pmatrix} p^2-\mu & 0 \\ 0 & \mu - p^2\end{pmatrix}$ is the H_Δ from above with $\Delta = 0$. Thus

$$\operatorname{Tr}\left[\frac{2f'(\Gamma_0)}{2\Gamma_0-1}G^2\right] = 2|\varphi|^2 \operatorname{Tr}\left[\frac{\frac{H_0}{2T}}{\tanh\frac{H_0}{2T}}\right] = 4|\varphi|^2\frac{\frac{E_0}{2T}}{\tanh\frac{E_0}{2T}} = \frac{2}{T}K_T^0|\varphi|^2.$$

since $\frac{x}{\tanh x}$ is even and $H_0^2 = E_0^2$. Thus, formally we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} S(\Gamma_0 + tG) = -\int \frac{2}{T}K_T^0(p)|\varphi(p)|^2\,\mathrm{d}p = -\frac{2}{T}\left\langle\varphi\middle|K_T^0\middle|\varphi\right\rangle.$$

To pull the differentiation inside the integral in the first equality above we use a dominated convergence argument, using that φ is of compact support. In total we conclude that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} \mathcal{F}(\Gamma_0 + tG) = 2\left\langle\varphi\middle|K_T^0 + V\middle|\varphi\right\rangle.$$

Since this operator has a negative eigenvalue, we may choose $\varphi \in C_0^\infty$ such that this is negative. We conclude that Γ_0 is not the minimum, and so (i) holds.

In the case $T = 0$ the above argument simplifies (due to the non-existence of the entropy term) and we arrive at

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} \mathcal{F}(\Gamma_0 + tG) = 2\left\langle\varphi|V|\varphi\right\rangle \le 2\left\langle\varphi\middle|K_0^0 + V\middle|\varphi\right\rangle.$$

And the same argument as above leads to the conclusion from here. □

Remark 3.9. The proof above shows in fact the stronger statement, that if either (hence all) of the equivalent conditions are satisfied, then Γ_0 is not even a local minimum, so that a system in this state would not be stable.

4 Asymptotics of the Critical Temperature

In this section we will consider our system in different asymptotic limits and find formulas for the critical temperature in those. We first consider the limit of weak coupling.

4.1 Weak Coupling

In this section we will consider a system in the weak coupling limit, meaning that V is small. The way we do this formally is to introduce a coupling constant $\lambda > 0$, and consider the potential λV for fixed V and consider the limit $\lambda \searrow 0$. This section is based on [9].

Assumption 4.1. We assume that $V \in L^1 \cap L^{3/2}$ is real-valued and reflection-symmetric.

From now on we will forget the superscript 0 on K_T^0, and just write K_T. The critical temperature is thus $T_c(\lambda V) = \inf\{T \geq 0 : K_T + \lambda V \geq 0\}$. Whether $T_c > 0$ or $T_c = 0$ for a given λV is non-trivial. In theorem 4.3 below we give a sufficient condition for $T_c > 0$.

The critical temperature $T_c(\lambda V)$ is characterized by 0 being the lowest eigenvalue of $K_T + \lambda V$, exactly for $T = T_c$. We use the Birman-Schwinger principle [20, pp. 75-77] to translate this property to the property that

$$B_T = \lambda V^{1/2} K_T^{-1} |V|^{1/2} = \lambda J |V|^{1/2} K_T^{-1} |V|^{1/2}$$

has -1 as its smallest eigenvalue. Here $V^{1/2}$ is defined by $V^{1/2} := \operatorname{sgn} V |V|^{1/2}$ and $J := \operatorname{sgn} V$. More precisely

Proposition 4.2 ([9, Lem. 1])**.** *Let $T > 0$. Then B_T is Hilbert-Schmidt and has real spectrum. Moreover,*

- *if $T_c > 0$ then B_T has -1 as the lowest eigenvalue exactly for $T = T_c$,*
- *if $T_c = 0$ then B_T has spectrum contained in $(-1, \infty)$ for all $T > 0$.*

Proof. First we have that $K_T \geq cp^2$ hence $K_T^{-1} \leq Cp^{-2}$. This function, seen as a tempered distribution, has Fourier transform $C\frac{1}{|x|}$. To see this note that $\frac{1}{p^2}$ is (-2)-homogeneous, and so its Fourier transform will be $(-3-(-2))$-homogeneous, i.e. (-1)-homogeneous. Additionally $\frac{1}{p^2}$ is rotation-invariant, so its Fourier transform must be as well. In total the Fourier transform must be $C\frac{1}{|x|}$ for some constant C.

Hence by the Hardy-Littlewood-Sobolev inequality [19, Thm 4.3] we thus see that the operator $P_T := \lambda |V|^{1/2} K_T^{-1} |V|^{1/2}$ is Hilbert-Schmidt. Thus $B_T = JP_T$ is. Moreover, since B_T and $P_T^{1/2} J P_T^{1/2}$ are isospectral up to 0 being a spectral value, and $P_T^{1/2} J P_T^{1/2}$ is self-adjoint, we get that B_T has real spectrum. (In fact 0 is in both spectra, since both operators are Hilbert-Schmidt.)

For $T_c > 0$ we have that T_c is characterised exactly by $K_T + \lambda V$ having 0 as its lowest eigenvalue. Now, if ψ is an eigenvector for $K_T + \lambda V$ with eigenvalue 0 then $-\psi = \lambda K_T^{-1} V \psi$. Hence $\varphi := V^{1/2}\psi$ is an eigenvector of B_T with eigenvalue -1. This shows that for $T = T_c$ we have that B_T has -1 as an eigenvalue. (Note that $K_T + \lambda V$ as a quadratic form has domain H^1 since $K_T \sim p^2$ for large p and by the Hardy-Littlewood-Sobolev inequality. Also $\psi \in H^1$ and so $\varphi \in L^2$ is well-defined.)

Suppose now for contradiction that B_T has -1 as an eigenvalue for some $T > T_c$, and let $\varphi \in L^2$ be an eigenvector. Define $\psi := -K_T^{-1}|V|^{1/2}\varphi$. One computes that distributionally $(K_T + \lambda V)\psi = 0$, and so that $K_T + \lambda V$ has 0 as an eigenvalue. Now, K_T is strictly increasing in T and so by lowering T while still keeping $T > T_c$ we get that $K_T + \lambda V$ has a negative eigenvalue for some $T > T_c$. Contradiction. We conclude that -1 is not an eigenvalue for B_T for any $T > T_c$.

We now show that -1 is the lowest eigenvalue of B_{T_c}. First, we claim that the eigenvalues of B_T depend continuously on T. This follows since we can bound by Taylor expansion in T

$$0 \leq \left|K_{T'}^{-1}(p) - K_T^{-1}(p)\right| \leq \left(1 - \tanh^2 \frac{p^2-\mu}{2T}\right) \frac{|T'-T|}{2T^2} =: L_{T,T'}(p) \in L^1$$

pointwise. And so $\lambda |V|^{1/2} L_{T,T_0} |V|^{1/2}$ is a bounded operator $L^2 \to L^2$ with vanishing norm in the limit $T \to T_0$. This is seen by the following decomposition of the operator.

$$L^2 \xrightarrow{|V|^{1/2}} L^1 \xrightarrow{\hat{\cdot}} L^\infty \xrightarrow{L_{T,T'}} L^1 \xrightarrow{\check{\cdot}} L^\infty \xrightarrow{|V|^{1/2}} L^2 \ ,$$

where $\hat{\cdot}, \check{\cdot} : L^1 \to L^\infty$ denote the Fourier transform and its adjoint. It follows that, in the limit $T' \to T$ we have $\langle \psi | B_{T'} - B_T | \psi \rangle = o(1)$ uniformly in bounded ψ. Thus by the min-max principle [19, Thm. 12.1] we get that the eigenvalues are continuous as desired.

Moreover, as $T \to \infty$ we get that $K_T^{-1} \to 0$ pointwise and so the eigenvalues of B_T converge to 0 in this limit too. Hence, suppose for contradiction that -1 is not the lowest eigenvalue for B_{T_c}. Increasing the temperature the lowest eigenvalue would then eventually become -1 for some $T > T_c$ contradicting the previous. We conclude that B_T has -1 as its lowest eigenvalue exactly for $T = T_c$.

The above argument also gives the $T_c = 0$ case. □

Define for $\mu > 0$ the operator

$$\mathcal{V}_\mu : L^2(\Omega_\mu) \to L^2(\Omega_\mu), \qquad \mathcal{V}_\mu u(p) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{\mu}} \int_{\Omega_\mu} \hat{V}(p-q) u(q) \, \mathrm{d}\omega(q),$$

where Ω_μ is the Fermi sphere, the sphere in momentum-space of radius $\sqrt{\mu}$ and $\mathrm{d}\omega$ denotes integration w.r.t. to the Lebesgue measure normalised with $|\Omega_\mu| = 4\pi\mu$. Since $V \in L^1$, its Fourier transform $\hat{V}$ is continuous by the Riemann-Lebesgue lemma [8, Thm. 8.22], and so its restriction to a null-set (here translates of the Fermi sphere) is sensible, thus this $\mathcal{V}_\mu$ is well-defined.

To see that $\mathcal{V}_\mu$ is self-adjoint define $\mathfrak{F} : L^1 \to L^2(\Omega_\mu)$ to be the Fourier transform restricted to the Fermi sphere. Again, by the Riemann-Lebesgue lemma, this operator is well-defined and bounded. Since $V \in L^1$ we have that $\mathfrak{F}|V|^{1/2} : L^2 \to L^2(\Omega_\mu)$ is bounded and so, with $J = \operatorname{sgn} V$ we have $\mathcal{V}_\mu = \frac{1}{\sqrt{\mu}} \left(\mathfrak{F}|V|^{1/2}\right) J \left(\mathfrak{F}|V|^{1/2}\right)^\dagger$. Since $J = \operatorname{sgn} V$ is real-valued, we have that $\mathcal{V}_\mu$ is self-adjoint. Define now

$$e_\mu(V) := \inf \operatorname{spec} \mathcal{V}_\mu.$$

The integral kernel of $\mathcal{V}_\mu$ is $\frac{1}{(2\pi)^{3/2}\sqrt{\mu}} \hat{V}(p-q) \in L^2(\Omega_\mu \times \Omega_\mu)$ by the Riemann-Lebesgue lemma [8, Thm. 8.22]. Hence $\mathcal{V}_\mu$ is Hilbert-Schmidt. Thus its eigenvalues converge to 0. In particular we have $e_\mu(V) \leq 0$. In fact, $\mathcal{V}_\mu$ is even trace-class (this will be shown below) and its trace equals $\frac{\sqrt{\mu}}{2\pi^2} \int V \, \mathrm{d}x$. In particular if $\int V \, \mathrm{d}x < 0$ we have that $e_\mu(V) < 0$. We prove the following theorem.

Theorem 4.3 ([9, Thm. 1])**.** *Let $V \in L^{3/2} \cap L^1$ be real-valued and reflection-symmetric and let $\mu > 0$. Suppose that $e_\mu(V) < 0$. Then $T_c(\lambda V) > 0$ for all $\lambda > 0$ and*

$$\lim_{\lambda \to 0} \lambda \log \frac{\mu}{T_c(\lambda V)} = -\frac{1}{e_\mu(V)}$$

That is, for small λ the critical temperature is $T_c \sim \mu \exp\left(\frac{-1}{\lambda e_\mu}\right)$.

Remark 4.4. In the case of a radial potential $V(x) = V(|x|)$ the eigenfunctions are (scaled versions of) spherical harmonics and the eigenvalues are given by

$$\frac{\sqrt{\mu}}{2\pi^2} \int V(x) j_\ell(\sqrt{\mu}|x|)^2 \, \mathrm{d}x, \qquad \ell \in \mathbb{N}_0.$$

Here j_ℓ is the spherical Bessel function of the first kind. To see this we simply calculate $\mathcal{V}_\mu Y_\ell^m$ for a (scaled) spherical harmonic $Y_\ell^m\left(\frac{p}{\sqrt{\mu}}\right)$. I thank fellow student Benjamin Tangen Søgaard for this computation. We use the plane wave expansion ([12, p. 406] and addition formula)

$$e^{ikx} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^\ell j_\ell(|k||x|) Y_\ell^m(\hat{k}) \overline{Y_\ell^m(\hat{x})},$$

where $\hat{\circ}$ denotes the unit vector in direction $\circ$, i.e. $\hat{x} = x/|x|$. We compute

$$\begin{aligned}\mathcal{V}_\mu Y_\ell^m(\hat{p}) &= \frac{1}{(2\pi)^3}\frac{1}{\sqrt{\mu}}\int_{\mathbb{R}^3}\int_{\Omega_\mu} V(x)e^{-ix(p-q)}Y_\ell^m(\hat{q})\,\mathrm{d}\omega(q)\,\mathrm{d}x \\ &= \frac{2\sqrt{\mu}}{\pi}\sum_{\ell_1,\ell_2}\sum_{m_1,m_2}\int_0^\infty\int_{S^2}\int_{S^2} V(|x|)(-i)^{\ell_1}i^{\ell_2}j_{\ell_1}(\sqrt{\mu}|x|)j_{\ell_2}(\sqrt{\mu}|x|) \\ &\qquad\times\overline{Y_{\ell_1}^{m_1}(\hat{p})}Y_{\ell_1}^{m_1}(\hat{x})Y_{\ell_2}^{m_2}(q)\overline{Y_{\ell_2}^{m_2}(\hat{x})}Y_\ell^m(q)\,\mathrm{d}\omega(q)\,\mathrm{d}\omega(\hat{x})\,\mathrm{d}|x|,\end{aligned}$$

where we computed the x-integral in spherical coordinates, scaled the q-integral and used the plane wave expansion as above. Using that $\overline{Y_\ell^m} = (-1)^m Y_\ell^{-m}$ and the orthonormality relations for the spherical harmonics we arrive at

$$\mathcal{V}_\mu Y_\ell^m(p) = \frac{2\sqrt{\mu}}{\pi}\int_0^\infty V(|x|)j_\ell(\sqrt{\mu}|x|)^2 Y_\ell^m(\hat{p})\,\mathrm{d}|x| = \left[\frac{\sqrt{\mu}}{2\pi^2}\int_{\mathbb{R}^3} V(x)j_\ell(\sqrt{\mu}|x|)^2\,\mathrm{d}x\right]Y_\ell^m(\hat{p})$$

as desired. Since there are $2\ell - 1$ such spherical harmonics which are ℓ-homogeneous, we see that these eigenvalues are $(2\ell-1)$-fold degenerate. In particular for $\ell = 0$, i.e. for the constant function, the eigenvalue is non-degenerate. Since the functions Y_ℓ^m span all of $L^2(S^2)$ these are all the eigenvalues.

If in addition the Fourier transform is non-positive, $\hat{V} \leq 0$ then for any u we have

$$\begin{aligned}(2\pi)^{3/2}\sqrt{\mu}\left\langle u\middle|\mathcal{V}_\mu\middle|u\right\rangle &= \iint_{\Omega_\mu\times\Omega_\mu} u(p)\hat{V}(p-q)u(q)\,\mathrm{d}\omega(q)\,\mathrm{d}\omega(p) \\ &\geq \iint_{\Omega_\mu\times\Omega_\mu} |u(p)|\hat{V}(p-q)|u(q)|\,\mathrm{d}\omega(q)\,\mathrm{d}\omega(p).\end{aligned}$$

Thus for the ground state we may take u non-negative. Hence it is not orthogonal to the constant function and so the ground state must be the constant function, since the $\ell = 0$ eigenvalue is non-degenerate. That is, the ground state is $u = \frac{1}{\sqrt{4\pi\mu}}$ in this case. This will be useful in section 5, when we consider such potentials. Specifically we will use this in the proof of lemma 5.7

We now prove the theorem. Define the operator $X : L^2 \to L^2$ by $X = \frac{1}{\sqrt{\mu}}|V|^{1/2}\mathfrak{F}^\dagger\mathfrak{F}|V|^{1/2}$, where by $|V|^{1/2}\mathfrak{F}^\dagger$ we mean the adjoint of $\mathfrak{F}|V|^{1/2}$. Then X is non-negative and has integral kernel

$$X(x,y) = |V(x)|^{1/2}\frac{1}{2\pi^2}\frac{\sin\sqrt{\mu}|x-y|}{|x-y|}|V(y)|^{1/2}.$$

This follows from the fact that $\int_{S^2} e^{ipx}\,\mathrm{d}\omega(p) = 4\pi\frac{\sin|x|}{|x|}$, which can be computed by choosing coordinates with x aligned with the polar angle being zero. We may compute $\operatorname{Tr} X = \frac{\sqrt{\mu}}{2\pi^2}\int |V(x)|\,\mathrm{d}x < \infty$. Thus X is trace-class.

Define also Y_T by

$$B_T = \lambda\log\left(1 + \frac{\mu}{2T}\right)JX + \lambda Y_T,$$

where again $J = \operatorname{sgn} V$, is the multiplication operator in configuration space, multiplying by the sign of V. It is immediate that the operator Y_T is Hilbert-Schmidt. In fact we have

Lemma 4.5 ([9, Lem. 2])**.** *For any $T > 0$ the operator Y_T is Hilbert-Schmidt. Moreover, its Hilbert-Schmidt norm is uniformly bounded in $T > 0$.*

This lemma is the key technical result needed to prove our theorem. We postpone its proof. To relate e_μ to B_T we have the following lemma

Lemma 4.6 ([9, Lem. 3])**.** *The operators JX (on L^2) and $\mathcal{V}_\mu$ (on $L^2(\Omega_\mu)$) have the same spectrum.*

Proof. We find two operators $A : L^2 \to L^2(\Omega_\mu)$ and $B : L^2(\Omega_\mu) \to L^2$ such that $AB = \mathcal{V}_\mu$ and $BA = JX$. Then the spectra of JX and $\mathcal{V}_\mu$ coincide, except possibly at 0. However, 0 is in both spectra, since both JX and $\mathcal{V}_\mu$ are Hilbert-Schmidt. Using that $\int_{S^2} e^{ipx}\,\mathrm{d}\omega(p) = 4\pi\frac{\sin|x|}{|x|}$ it is straightforward to check that the following choice of A and B work.

$$A\psi(p) = \frac{1}{(2\pi)^{3/2}}\int |V(x)|^{1/2}\psi(x)e^{-ipx}\,\mathrm{d}x, \quad Bu(x) = V(x)^{1/2}\frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{\mu}}\int_{\Omega_\mu} u(p)e^{ipx}\,\mathrm{d}\omega(p).$$

That is, $A = \mathfrak{F}|V|^{1/2}$ and $B = \frac{1}{\sqrt{\mu}}V^{1/2}\mathfrak{F}^\dagger$. □

Since X is trace-class we get that JX is and so $\mathcal{V}_\mu$ is. Moreover, $\operatorname{Tr}\mathcal{V}_\mu = \operatorname{Tr} JX = \frac{\sqrt{\mu}}{2\pi^2}\int V(x)\,\mathrm{d}x$ as previously claimed. We are now ready to prove the theorem

Proof of Theorem 4.3. We consider the spectrum of operators of the form $\alpha JX + \lambda Y_T$ with $\alpha > 0$. As in [9] we will use the notation $\Theta(t)$ for a term that grows of order t, i.e. if there exists constants $0 < c < C$ with $ct \le \Theta(t) \le Ct$.

For any $z \notin \operatorname{spec}\alpha JX$ we have that

$$\alpha JX + \lambda Y_T - z = (\alpha JX - z)\left(1 + \lambda(\alpha JX - z)^{-1}Y_T\right).$$

Hence this is invertible if

$$\lambda\,\|Y_T\|\left\|\frac{1}{\alpha JX - z}\right\| < 1.$$

Now,

$$\frac{1}{\alpha JX - z} = -\frac{1}{z} + \frac{\alpha}{z}JX^{1/2}\frac{1}{\alpha X^{1/2}JX^{1/2} - z}X^{1/2}$$

as can easily be checked. The operators $\alpha X^{1/2}JX^{1/2}$ and αJX have the same spectrum, except possibly at 0. Since αJX is Hilbert-Schmidt, 0 is in the spectrum of αJX. By ordinary spectral calculus we have $\left\|\frac{1}{\alpha X^{1/2}JX^{1/2}-z}\right\| = \frac{1}{\operatorname{dist}(z,\operatorname{spec}(X^{1/2}JX^{1/2}))} \le \frac{1}{d}$ for z a distance at least d from the spectrum of αJX. We conclude that

$$\left\|\frac{1}{\alpha JX - z}\right\| \le \frac{1}{d} + \frac{\alpha}{d^2}\left\|X^{1/2}\right\|^2 = \frac{1}{d} + \frac{\alpha}{d^2}\|X\|$$

In particular z is not in the spectrum of $\alpha JX + \lambda Y_T$ if $d = \operatorname{dist}(z, \operatorname{spec}\alpha JX) \ge \Theta(\sqrt{\alpha\lambda}) + \Theta(\lambda)$ since $\|Y_T\| \le \|Y_T\|_2 \le \sup_{T>0}\|Y_T\|_2 < \infty$ by Lemma 4.5. Since Y_T is bounded we have that the spectrum of $\alpha JX + \lambda Y_T$ depend continuously on λ and is the same as the spectrum of αJX for $\lambda = 0$. Hence, as $\lambda \to 0$ there must be some eigenvalue approaching the lowest eigenvalue of JX. In total we conclude that the lowest eigenvalue of $\alpha JX + \lambda Y_T$ differs from that of αJX by a term of order at most $\Theta(\sqrt{\alpha\lambda}) + \Theta(\lambda)$.

Suppose now $e_\mu(V) < 0$. Then JX has a negative eigenvalue and so the spectrum of the Birman-Schwinger operator $B_T = \lambda\log\left(1 + \frac{\mu}{2T}\right)JX + \lambda Y_T$ becomes arbitrarily negative as $T \to 0$, since Y_T is bounded. In particular for some $T > 0$, B_T will have an eigenvalue of -1 and thus $T_c > 0$. Moreover, at $T = T_c$ we have that λY_{T_c} tends to 0 in the limit $\lambda \to 0$, hence the term $\alpha = \lambda\log\left(1 + \frac{\mu}{2T_c(\lambda V)}\right)$ is $\Theta(1)$ as $\lambda \to 0$.

Let now $\alpha = \lambda\log\left(1 + \frac{\mu}{2T}\right)$. Then the lowest eigenvalue of B_T differs from $\lambda\log\left(1 + \frac{\mu}{2T}\right)e_\mu(V)$ by terms of order at most $\Theta(\sqrt{\alpha\lambda}) + \Theta(\lambda)$. At T_c we have $\alpha = \Theta(1)$ as $\lambda \to 0$. Also λ vanishes faster than $\sqrt{\lambda}$. Hence, since the lowest eigenvalue here is -1, we have $-1 = \lambda\log\left(1 + \frac{\mu}{2T}\right)e_\mu(V) + O(\sqrt{\lambda})$ in the limit $\lambda \to 0$. This gives the desired asymptotics. □

We now give the proof of Lemma 4.5.

Proof of Lemma 4.5. By scaling we may wlog assume $\mu = 1$. The idea is to decompose K_T^{-1} into different terms, all of which are bounded except for the terms leading to the JX-term in the expression for B_T.

Define the function $g(t) = \frac{t(1+e^{-t})}{(t+2)(1-e^{-t})}$. Note that then $K_T(p) = \left(|p^2-1| + 2T\right) g\left(\frac{|p^2-1|}{T}\right)$. Decompose K_T^{-1} as $K_T^{-1} = L_T^{(1)} + M_T^{(1)}$ where $L_T^{(1)} = 1_{\{p^2<2\}} K_T^{-1}$ and $M_T^{(1)} = 1_{\{p^2 \geq 2\}} K_T^{-1}$. Since $\inf_t g(t) > 0$ we have that $M_T^{(1)} \leq C\frac{1}{p^2}$. Hence we may bound the kernel of $|V|^{1/2} M_T^{(1)} |V|^{1/2}$ by $|V(x)|^{1/2} \frac{1}{|x-y|} |V(y)|^{1/2} \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ by the Hardy-Littlewood-Sobolev inequality [19, Thm. 4.3]. Here we again used that $\frac{1}{p^2}$, seen as a tempered distribution, has Fourier transform $\frac{1}{|x|}$ up to some constant. We conclude that $\left\| V^{1/2} M_T^{(1)} |V|^{1/2} \right\|_2$ is bounded uniformly in T.

Now, the integral kernel of $L_T^{(1)}$ is

$$
\begin{aligned}
L_T^{(1)}(x,y) &= \frac{1}{(2\pi)^3} \int_{\{p^2<2\}} \frac{e^{ip(x-y)}}{(|p^2-1| + 2T) g(|p^2-1|/T)} \, \mathrm{d}p \\
&= \frac{1}{2\pi^2} \int_0^{\sqrt{2}} \frac{k}{(|k^2-1| + 2T) g(|k^2-1|/T)} \frac{\sin k|x-y|}{|x-y|} \, \mathrm{d}k
\end{aligned}
$$

by computing the spherical part of the integral, using that $\int_{S^2} e^{ipx} \, \mathrm{d}\omega(p) = 4\pi \frac{\sin|x|}{|x|}$. Decompose now $L_T^{(1)}$ as $L_T^{(1)} = L_T^{(2)} + M_T^{(2)}$ with integral kernels

$$
\begin{aligned}
L_T^{(2)}(x,y) &= \frac{1}{2\pi^2} \int_0^{\sqrt{2}} \frac{k}{|k^2-1| + 2T} \frac{\sin k|x-y|}{|x-y|} \, \mathrm{d}k, \\
M_T^{(2)}(x,y) &= \frac{1}{2\pi^2} \int_0^{\sqrt{2}} \frac{k}{|k^2-1| + 2T} \frac{\sin k|x-y|}{|x-y|} \left(\frac{1}{g(|k^2-1|/T)} - 1 \right) \mathrm{d}k.
\end{aligned}
$$

Bounding $|\sin k|x-y|| \leq k|x-y| \leq \sqrt{2}|x-y|$ and using the substitution $t = |k^2-1|/T$ we get

$$
\left| M_T^{(2)}(x,y) \right| \leq \frac{\sqrt{2}}{2\pi^2} \int_0^{1/T} \frac{1}{t+2} \left(\frac{1}{g(t)} - 1 \right) \mathrm{d}t.
$$

Now, as $t \to \infty$ we have that $\frac{1}{g(t)} - 1 = \frac{2}{t} + O(e^{-t})$. We get that the kernel $M_T^{(2)}(x,y)$ is bounded uniformly in T. Hence we may bound

$$
\left\| V^{1/2} M_T^{(2)} |V|^{1/2} \right\|_2 \leq \|V\|_1 \sup_{x,y} |M_T^{(2)}(x,y)| < \infty
$$

uniformly in T. Decomposing finally $L_T^{(2)} = L_T^{(3)} + M_T^{(3)}$ with kernels

$$
\begin{aligned}
L_T^{(3)}(x,y) &= \frac{1}{2\pi^2} \frac{\sin|x-y|}{|x-y|} \int_0^{\sqrt{2}} \frac{k}{|k^2-1| + 2T} \, \mathrm{d}k = \frac{1}{2\pi^2} \log\left(1 + \frac{1}{2T}\right) \frac{\sin|x-y|}{|x-y|}, \\
M_T^{(3)}(x,y) &= \frac{1}{2\pi^2} \int_0^{\sqrt{2}} \frac{k}{|k^2-1| + 2T} \frac{\sin k|x-y| - \sin|x-y|}{|x-y|} \, \mathrm{d}k.
\end{aligned}
$$

Thus, $V^{1/2}L_T^{(3)}|V|^{1/2} = \log\left(1+\frac{1}{2T}\right) JX$ and so $Y_T = V^{1/2}\left(M_T^{(1)} + M_T^{(2)} + M_T^{(3)}\right)|V|^{1/2}$. Since sin is Lipschitz with constant 1 we get that $|\sin k|x-y| - \sin|x-y|| \leq |k-1||x-y|$ and so

$$\left|M_T^{(3)}(x,y)\right| \leq \frac{1}{2\pi^2}\int_0^{\sqrt{2}} \frac{k}{k+1+2T}\,\mathrm{d}k \leq \frac{1}{2\pi^2}\int_0^{\sqrt{2}} \frac{k}{k+1}\,\mathrm{d}k < \infty$$

Thus we get $\left\|V^{1/2}M_T^{(3)}|V|^{1/2}\right\|_2$ is bounded uniformly in T as before. Hence, Y_T is uniformly bounded as desired. □

4.2 Refined Weak Coupling

In this section we consider again the limit of weak coupling, only now we give more precise asymptotic results for the critical temperature T_c. This section is based on [13].

We will again work with the assumption that $V \in L^1 \cap L^{3/2}$ is real-valued and reflection-symmetric. Define the operator $\mathcal{W}_\mu : L^2(\Omega_\mu) \to L^2(\Omega_\mu)$ for $\mu > 0$ by

$$\langle u|\mathcal{W}_\mu|u\rangle = \int_0^\infty \frac{p^2}{|p^2-\mu|}\int_{S^2} |\psi(p)|^2 - \left|\psi(\sqrt{\mu}\hat{p})\right|^2 \mathrm{d}\omega(\hat{p}) + \int_{S^2}\left|\psi(\sqrt{\mu}\hat{p})\right|^2 \mathrm{d}\omega(\hat{p})\,\mathrm{d}|p|$$

where $\psi(p) = \frac{1}{(2\pi)^{3/2}}\int_{\Omega_\mu}\hat{V}(p-q)u(q)\,\mathrm{d}\omega(q)$ and we again denote by $\hat{p}$ the vector in direction p with unit length, i.e. $\hat{p} = p/|p|$. We of course need to argue that this integral is indeed finite. This follows by the fact that the spherical integral $\int_{S^2}|\psi(p)|^2\,\mathrm{d}\omega(\hat{p})$ is a Lipschitz continuous function of $|p|$, which we now show. We have

$$\begin{aligned}
&\int_{S^2}|\psi(p)|^2\,\mathrm{d}\omega(\hat{p})\\
&= \int_{S^2}\frac{1}{(2\pi)^6}\iint V(y)V(x)e^{-ipx}e^{ipy}\int_{\Omega_\mu} e^{iqx}u(q)\,\mathrm{d}\omega(q)\int_{\Omega_\mu} e^{-iry}\overline{u(r)}\,\mathrm{d}\omega(r)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\omega(\hat{p})\\
&= \frac{4\pi}{(2\pi)^6}\iint \frac{\sin|p||x-y|}{|p||x-y|}V(x)V(y)\int_{\Omega_\mu} e^{iqx}u(q)\,\mathrm{d}\omega(q)\int_{\Omega_\mu} e^{-iry}\overline{u(r)}\,\mathrm{d}\omega(r)\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

since $\int_{S^2} e^{ip(y-x)}\,\mathrm{d}\omega(\hat{p}) = 4\pi\frac{\sin|p||x-y|}{|p||x-y|}$. We thus get

$$\begin{aligned}
&\left|\int_{S^2}|\psi(p)|^2\,\mathrm{d}\omega(\hat{p}) - \int_{S^2}|\psi(p')|^2\,\mathrm{d}\omega(\hat{p}')\right|\\
&\qquad \leq C\iint \frac{||p|-|p'||}{|p|+|p'|}|V(x)||V(y)|\left(\int_{\Omega_\mu}|u(q)|\,\mathrm{d}\omega(q)\right)^2\mathrm{d}x\,\mathrm{d}y \leq C\frac{||p|-|p'||}{|p|+|p'|}
\end{aligned}$$

since $\left|\frac{\sin a}{a} - \frac{\sin b}{b}\right| \leq C\frac{|a-b|}{a+b}$ for all $a,b>0$. We conclude that the spherical integral $\int_{S^2}|\psi(p)|^2\,\mathrm{d}\omega(\hat{p})$ is a Lipschitz continuous function of $|p|$ at every $p \neq 0$. In particular at $|p| = \sqrt{\mu}$. Hence this shows that the defining integral for $\mathcal{W}_\mu$ is well-defined and finite. Thus $\mathcal{W}_\mu$ is a bounded operator $L^2(\Omega_\mu) \to L^2(\Omega_\mu)$. We will in fact show that $\mathcal{W}_\mu$ is Hilbert-Schmidt in the proof of theorem 4.7 below.

Define the number

$$b_\mu(\lambda) := \inf\operatorname{spec}\left(\frac{\pi}{2\sqrt{\mu}}\lambda\mathcal{V}_\mu - \frac{\pi}{2\mu}\lambda^2\mathcal{W}_\mu\right).$$

Note that if $e_\mu = \inf\operatorname{spec}\mathcal{V}_\mu < 0$ then also $b_\mu(\lambda) < 0$ for sufficiently small λ. If also the lowest eigenvalue of $\mathcal{V}_\mu$ is non-degenerate, meaning that the eigenspace is one-dimensional and spanned by u, then

$$b_\mu(\lambda) = \left\langle u\middle|\frac{\pi}{2\sqrt{\mu}}\lambda\mathcal{V}_\mu - \frac{\pi}{2\mu}\lambda^2\mathcal{W}_\mu\middle|u\right\rangle + O(\lambda^3) = \lambda\frac{\pi e_\mu}{2\sqrt{\mu}} - \lambda^2\frac{\pi\left\langle u\middle|\mathcal{W}_\mu\middle|u\right\rangle}{2\mu} + O(\lambda^3).$$

This will follow from the proof below. We prove the following theorem.

Theorem 4.7 ([13, Thm. 1])**.** *Let $V \in L^1 \cap L^{3/2}$ be real-valued and reflection-symmetric and let $\mu > 0$. Assume that $e_\mu(V) = \inf\operatorname{spec}\mathcal{V}_\mu < 0$. Then the critical temperature T_c satisfies*

$$\lim_{\lambda\to 0}\left(\log\frac{\mu}{T_c(\lambda V)} + \frac{\pi}{2\sqrt{\mu}b_\mu(\lambda)}\right) = 2 - \gamma - \log\frac{8}{\pi}.$$

Here γ denotes the Euler-Mascheroni constant.

That is, in the limit $\lambda \to 0$ the critical temperature satisfies

$$T_c = \mu\left(\frac{8}{\pi}e^{\gamma-2} + o(1)\right)\exp\left(\frac{\pi}{2\sqrt{\mu}b_\mu(\lambda)}\right).$$

Proof. The idea is more or less again to identify the singular part of K_T^{-1}. Define

$$m_\mu(T) = \frac{1}{4\pi\mu}\int_{\mathbb{R}^3}\frac{1}{K_T(p)} - \frac{1}{p^2}\,\mathrm{d}p.$$

In lemma 4.8 below we show that as $T \to 0$ we have $m_\mu(T) = \frac{1}{\sqrt{\mu}}\left(\log\frac{\mu}{T} + 2 - \gamma + \log\frac{8}{\pi} + o(1)\right)$. Define the operator M_T by

$$M_T = K_T^{-1} - m_\mu(T)\mathfrak{F}^\dagger\mathfrak{F}.$$

Of course $\mathfrak{F}^\dagger$ on its own is nonsensical, however. By $\mathfrak{F}^\dagger\mathfrak{F}$ we will mean the operator with kernel $\frac{\sqrt{\mu}}{2\pi^2}\frac{\sin\sqrt{\mu}|x-y|}{|x-y|}$, motivated by the fact that $V^{1/2}\mathfrak{F}^\dagger\mathfrak{F}|V|^{1/2}$ has kernel $V(x)^{1/2}\frac{\sqrt{\mu}}{2\pi^2}\frac{\sin\sqrt{\mu}|x-y|}{|x-y|}|V(y)|^{1/2}$.

Using the asymptotics for $m_\mu(T)$ we see that $V^{1/2}M_T|V|^{1/2} = Y_T + O(1)JX$ uniformly in small T. Hence, uniformly in small T we have by lemma 4.5 that $V^{1/2}M_T|V|^{1/2}$ is bounded in Hilbert-Schmidt norm. Thus for λ sufficiently small we have that $1+\lambda V^{1/2}M_T|V|^{1/2}$ is invertible, Hence we may write

$$1 + B_T = (1 + \lambda V^{1/2}M_T|V|^{1/2})\left(1 + \frac{\lambda m_\mu(T)}{1 + \lambda V^{1/2}M_T|V|^{1/2}}V^{1/2}\mathfrak{F}^\dagger\mathfrak{F}|V|^{1/2}\right)$$

Since -1 is the lowest eigenvalue for B_T exactly for $T = T_c$ we may restate this as the operator

$$\mathfrak{F}|V|^{1/2}\frac{\lambda m_\mu(T)}{1 + \lambda V^{1/2}M_T|V|^{1/2}}V^{1/2}\mathfrak{F}^\dagger$$

having -1 as its lowest eigenvalue, since this operator is isospectral to the right-most operator above. More precisely we have that -1 is the lowest eigenvalue for B_T if and only if -1 is the lowest eigenvalue for the latter operator. Thus, let $T = T_c$, so that -1 is indeed the lowest eigenvalue.

Then, inverting the middle operator by a Neumann series we have

$$\inf\operatorname{spec}\left(m_\mu(T_c)\left(\lambda\sqrt{\mu}\mathcal{V}_\mu - \lambda^2\mathfrak{F}VM_{T_c}V\mathfrak{F}^\dagger + O(\lambda^3)\right)\right) = -1,$$

where the error-term is uniformly bounded in small T. Moreover, by first-order perturbation we have

$$m_\mu(T_c) + \frac{1}{\lambda\sqrt{\mu}\,\langle u|\mathcal{V}_\mu|u\rangle - \lambda^2\,\langle u|\mathfrak{F}VM_{T_c}V\mathfrak{F}^\dagger|u\rangle + O(\lambda^3)} = 0,$$

where u is the ground state of $\mathcal{V}_\mu$. (In case of degeneracy, we should pick the u minimising the λ^2-term, see [13].) The error-term is uniformly bounded in small T. This is a standard Feynman-Hellmann type argument.

This is the equation we use to get the asymptotic behaviour. For the behaviour of $\mathfrak{F}VM_{T_c}V\mathfrak{F}^\dagger$ in this limit we again write $\psi(p) = \frac{1}{(2\pi)^{3/2}}\int_{\Omega_\mu}\hat{V}(p-q)u(q)\,\mathrm{d}\omega(q)$. We then have

$$\begin{aligned}\langle u|\mathfrak{F}VM_TV\mathfrak{F}^\dagger|u\rangle &= \int \frac{1}{K_T(p)}|\psi(p)|^2\,\mathrm{d}p - m_\mu(T)\int_{\Omega_\mu}|\psi(p)|^2\,\mathrm{d}\omega(p)\\ &= \int \frac{1}{K_T(p)}\left(|\psi(p)|^2 - |\psi(\sqrt{\mu}\hat{p})|^2\right) + \frac{1}{p^2}|\psi(\sqrt{\mu}\hat{p})|^2\,\mathrm{d}p.\end{aligned}$$

The integrand converges pointwise to the integrand in $\langle u|\mathcal{W}_\mu|u\rangle$. To see that we may interchange the limit $T \to 0$ and the integral, note that K_T decreases to $|p^2-\mu|$ for $T \to 0$. Hence by dominated convergence we get the desired. We conclude that

$$\lim_{\lambda\to 0}\left(m_\mu(T_c) + \frac{1}{\langle u|\lambda\sqrt{\mu}\mathcal{V}_\mu - \lambda^2\mathcal{W}_\mu|u\rangle}\right) = 0.$$

Thus,

$$\lim_{\lambda\to 0}\left(m_\mu(T_c) + \frac{1}{\inf\operatorname{spec}(\lambda\sqrt{\mu}\mathcal{V}_\mu - \lambda^2\mathcal{W}_\mu)}\right) = 0.$$

Now, by the definition of $b_\mu(\lambda)$ and the asymptotics of $m_\mu(T)$, lemma 4.8, we thus get

$$\lim_{\lambda\to 0}\left(\frac{1}{\sqrt{\mu}}\left(\log\frac{\mu}{T_c(\lambda V)} + \gamma - 2 + \log\frac{8}{\pi} + o(1)\right) + \frac{\pi}{2\mu b_\mu(\lambda)}\right) = 0,$$

which show the desired. □

We now prove the needed asymptotics for $m_\mu(T)$.

Lemma 4.8 ([13, Lem. 1])**.** *In the limit $T/\mu \to 0$ we have the following asymptotics*

$$m_\mu(T) = \frac{1}{4\pi\mu}\int_{\mathbb{R}^3}\frac{1}{K_T} - \frac{1}{p^2}\,\mathrm{d}p = \frac{1}{\sqrt{\mu}}\left(\log\frac{\mu}{T} + \gamma - 2 + \log\frac{8}{\pi} + o(1)\right),$$

where γ is the Euler-Mascheroni constant.

We state this result in a manner which also allows μ to vary. This we will need in section 4.3. Note that for constant μ, this formula hold in the limit $T \to 0$. The proof can be found in [13, Lem. 1]. We reproduce it here for convenience.

Proof. First, by rotational invariance we have

$$m_\mu(T) = \frac{1}{\mu}\int_0^{\sqrt{\mu}}\frac{1}{K_T} - \frac{1}{p^2}\,\mathrm{d}|p| + \frac{1}{\mu}\int_{\sqrt{\mu}}^\infty\frac{1}{K_T} - \frac{1}{p^2}\,\mathrm{d}|p|$$

Using the substitutions $t = \frac{\mu - p^2}{\mu}$ for the first integral and $t = \frac{p^2-\mu}{\mu}$ for the second we arrive at

$$m_\mu(T) = \frac{1}{2\sqrt{\mu}} \int_0^1 \frac{\sqrt{1-t}\tanh\left(\frac{\mu}{T}\frac{t}{2}\right)}{t} - \frac{1}{\sqrt{1-t}}\,\mathrm{d}t + \frac{1}{2\sqrt{\mu}} \int_0^\infty \frac{\sqrt{1+t}\tanh\left(\frac{\mu}{T}\frac{t}{2}\right)}{t} - \frac{1}{\sqrt{1+t}}\,\mathrm{d}t$$

Note that $\frac{1}{2} - \frac{1}{2}\tanh\frac{x}{2} = \frac{1}{1+e^x}$ and so for the integral from 1 to ∞ we have that the integrand can be dominated

$$\left|\frac{\sqrt{1+t}\tanh\frac{\mu}{T}\frac{t}{2}}{t} - \frac{1}{\sqrt{1+t}}\right| = \left|\frac{1}{t\sqrt{1+t}} - \frac{2\sqrt{1+t}}{t(1+e^{\mu t/T})}\right| \leq \frac{1}{t\sqrt{1+t}} + \frac{2\sqrt{1+t}}{t}\frac{1}{1+e^t}$$

uniformly in $T/\mu < 1$. Hence by dominated convergence we get

$$\lim_{T/\mu\to 0} \int_1^\infty \frac{\sqrt{1+t}\tanh\left(\frac{\mu}{T}\frac{t}{2}\right)}{t} - \frac{1}{\sqrt{1+t}}\,\mathrm{d}t = \int_1^\infty \frac{\sqrt{1+t}}{t} - \frac{1}{\sqrt{1+t}}\,\mathrm{d}t = 2\log(1+\sqrt{2}).$$

Similarly by dominated convergence we get

$$\lim_{T/\mu\to 0} \int_0^1 \frac{\sqrt{1\pm t}-1}{t}\tanh\left(\frac{\mu}{T}\frac{t}{2}\right)\,\mathrm{d}t = \int_0^1 \frac{\sqrt{1\pm t}-1}{t}\,\mathrm{d}t$$

And we compute

$$\int_0^1 \frac{\sqrt{1+t}-1}{t}\,\mathrm{d}t = 2\log 2 - 2 + 2\sqrt{2} - 2\log\left(1+\sqrt{2}\right),$$
$$\int_0^1 \frac{\sqrt{1-t}-1}{t}\,\mathrm{d}t = 2\log 2 - 2$$

and

$$\int_0^1 \frac{1}{\sqrt{1+t}} + \frac{1}{\sqrt{1-t}}\,\mathrm{d}t = 2\sqrt{2}.$$

Combining all this we get

$$m_\mu(T) = \frac{1}{\sqrt{\mu}}\left(\int_0^1 \frac{\tanh\left(\frac{\mu}{T}\frac{t}{2}\right)}{t}\,\mathrm{d}t + 2\log 2 - 2 + o(1)\right).$$

We now split the integral according to

$$\tanh\left(\frac{\mu}{T}\frac{t}{2}\right) = \left(1 - e^{-\mu t/T}\right) - e^{-\mu t/T}\tanh\left(\frac{\mu}{T}\frac{t}{2}\right).$$

Then by partial integration we have for the first part

$$\int_0^1 \frac{1-e^{-\mu t/T}}{t}\,\mathrm{d}t = \int_0^{\mu/T} \frac{1-e^{-t}}{t}\,\mathrm{d}t = \log\frac{\mu}{T}\left(1-e^{-\mu/T}\right) - \int_0^{\mu/T} e^{-s}\log t\,\mathrm{d}t.$$

hence [1, Eqn. 6.3.2]

$$\lim_{T/\mu\to 0}\left(\int_0^1 \frac{1-e^{-\mu t/T}}{t}\,\mathrm{d}t - \log\frac{\mu}{T}\right) = -\int_0^\infty e^{-t}\log t\,\mathrm{d}t = \gamma.$$

For the second part we have

$$\int_0^1 \frac{\tanh\left(\frac{\mu}{T}\frac{t}{2}\right)e^{-\mu t/T}}{t}\,\mathrm{d}t = \int_0^{\mu/T} \frac{\tanh\left(\frac{t}{2}\right)e^{-t}}{t}\,\mathrm{d}t = \int_0^\infty \frac{\tanh\left(\frac{t}{2}\right)e^{-t}}{t}\,\mathrm{d}t + o(1) = \log\frac{\pi}{2} + o(1)$$

as can be computed by Wolfram Alpha. Combining all this we get the desired. □

Computing all the integrals where we just state the result, can somewhat easily be done by hand with elementary integration techniques. We invite the interested reader to do the computation herself.

4.3 Low density

We now consider the limit of low density. Calling it a low density limit is maybe a bit too much. We will consider the limit $\mu \to 0$. This indeed corresponds to lowering the density. (This is easily seen for the normal states $\gamma_0(p) = (1 + \exp((p^2 - \mu)/T))^{-1}$.) However, if the interaction V is very attractive, one might even have a high density of particles for μ very small. Either way, we consider the critical temperature in the limit $\mu \to 0$. This section is based on [14].

Definition 4.9. Suppose -1 is not in the spectrum of $V^{1/2}\frac{1}{p^2}|V|^{1/2}$. Then the scattering length a of $2V$ is

$$a = a(V) := \frac{1}{4\pi}\left\langle |V|^{1/2} \middle| \frac{1}{1 + V^{1/2}\frac{1}{p^2}|V|^{1/2}} \middle| V^{1/2} \right\rangle.$$

We prove the following theorem

Theorem 4.10. *Suppose that $V(x)(1+|x|) \in L^1 \cap L^{3/2}$ is real-valued and reflection-symmetric, that the spectrum of $V^{1/2}\frac{1}{p^2}|V|^{1/2}$ is contained in $(-1,\infty)$ and that the scattering length $a < 0$ is negative. Then the critical temperature satisfies*

$$\lim_{\mu\to 0}\left(\log\frac{\mu}{T_c} + \frac{\pi}{2\sqrt{\mu}a}\right) = 2 - \gamma - \log\frac{8}{\pi},$$

where γ is the Euler-Mascheroni constant.

Put differently, in the limit $\mu \to 0$ we have

$$T_c = \mu\left(\frac{8}{\pi}e^{\gamma-2} + o(1)\right)\exp\left(\frac{\pi}{2\sqrt{\mu}a}\right).$$

The rest of this section is devoted to proving this theorem. First we prove that the critical temperature is sufficiently small

Proposition 4.11. *The critical temperature satisfies $T_c = o(\mu)$ in the limit $\mu \to 0$.*

Remark 4.12. This is not proved in the original paper [14]. There it is only proved that $T_c = O(\mu)$. We in fact need this stronger statement to prove proposition 4.13 below. The statement of proposition 4.13 in [14, Lem. 1] is not true. For large values of $T = O(\mu)$, the function $\tilde{m}_\mu(T)$ might be negative.

Proof. Let $\frac{-1}{\lambda}$ denote the smallest eigenvalue of $V^{1/2}\frac{1}{p^2}|V|^{1/2}$. Then $\lambda > 1$. Then the Birman-Schwinger principle implies that $p^2 + \lambda V \geq 0$. We may bound $\tanh t \leq \min\{1, t\}$ for $t \geq 0$ and so

$$K_T + V \geq \frac{1}{\lambda}|p^2 - \mu| + \left(1 - \frac{1}{\lambda}\right)2T + V \geq \frac{p^2 - \mu + (\lambda-1)2T + \lambda V}{\lambda} \geq \frac{-\mu + 2T(\lambda-1)}{\lambda}.$$

At $T < T_c$ the operator $K_T + V$ has a negative eigenvalue. Hence we conclude that $T_c \leq C\mu$ for the constant $C = \frac{1}{2(\lambda-1)} > 0$. Now, decompose the Birman-Schwinger operator B_T as

$$B_T := V^{1/2}\frac{1}{K_T}|V|^{1/2} = V^{1/2}\frac{1}{p^2}|V|^{1/2} + \tilde{m}_\mu(T)\left|V^{1/2}\right\rangle\left\langle|V|^{1/2}\right| + A_{T,\mu}.$$

Here $\tilde{m}_\mu(T) = \frac{\mu}{2\pi^2}m_\mu(T) = \frac{1}{(2\pi)^3}\int \frac{1}{K_T} - \frac{1}{p^2}\,\mathrm{d}p$ and $A_{T,\mu}$ is defined such that this holds. Let $c > 0$ be any constant. We now show, uniformly in T with $c\mu \leq T \leq C\mu$, that the second and third summand vanish in the limit $\mu \to 0$. It then follows that the spectrum of B_T approaches that of $V^{1/2}\frac{1}{p^2}|V|^{1/2}$.

Since the latter is contained in $\left[\frac{-1}{\lambda}, \infty\right)$, we get that $T > T_c$ for any such T. Thus $T_c = o(\mu)$ as desired. In the proof of lemma 4.8 we saw that $m_\mu(T)$ is of order $\frac{1}{\sqrt{\mu}}$ for T/μ bounded. Thus $\tilde{m}_\mu(T)$ vanishes and so the second term vanishes in the limit $\mu \to 0$. For the third term, its kernel is given by

$$A_{T,\mu}(x,y) = V(x)^{1/2}|V(y)|^{1/2}\frac{1}{2\pi^2}\int_0^\infty \left(\frac{\sin|p||x-y|}{|p||x-y|} - 1\right)\left(\frac{1}{K_T} - \frac{1}{p^2}\right) p^2 \,\mathrm{d}|p|,$$

where we used that $\int_{S^2} e^{ipx}\,\mathrm{d}\omega(\hat{p}) = 4\pi\frac{\sin|p||x-y|}{|p||x-y|}$. We now use the bound $\left|\frac{\sin b}{b} - 1\right| \le Cb^\alpha$ for any $0 \le \alpha \le 2$. We use this with $\alpha = \frac{1}{2}$. Thus,

$$|A_{T,\mu}(x,y)| \le C|V(x)|^{1/2}|V(y)|^{1/2}|x-y|^{1/2}\left[\int_0^{\sqrt{2\mu}}\left|\frac{1}{K_T} - \frac{1}{p^2}\right| p^{5/2}\,\mathrm{d}p + \int_{\sqrt{2\mu}}^\infty \left|\frac{1}{K_T} - \frac{1}{p^2}\right| p^{5/2}\,\mathrm{d}p\right].$$

Now, for the first integral, we may bound $K_T \ge 2T \ge 2c\mu$ so $\frac{1}{K_T} \le C\frac{1}{\mu}$ Thus

$$\int_0^{\sqrt{2\mu}}\left|\frac{1}{K_T} - \frac{1}{p^2}\right| p^{5/2}\,\mathrm{d}p \le \int_0^{\sqrt{2\mu}} C\frac{1}{\mu}\mu^{5/4} + C\mu^{1/4}\,\mathrm{d}p \le C\mu^{3/4}.$$

To bound the second integral we substitute $s = \frac{p^2-\mu}{\mu}$. Then

$$\begin{aligned}
\int_{\sqrt{2\mu}}^\infty \left|\frac{1}{K_T} - \frac{1}{p^2}\right| p^{5/2}\,\mathrm{d}p &= \frac{\mu^{3/4}}{2}\int_1^\infty \left|\frac{1+s}{s}\tanh\left(\frac{\mu s}{2T}\right) - 1\right| \frac{1}{(1+s)^{1/4}}\,\mathrm{d}s \\
&\le \frac{\mu^{3/4}}{2}\int_1^\infty \frac{\tanh\left(\frac{\mu s}{2T}\right)}{s(1+s)^{1/4}} + \frac{1-\tanh\left(\frac{\mu s}{2T}\right)}{(1+s)^{1/4}}\,\mathrm{d}s \\
&\le C\mu^{3/4} + \mu^{3/4}\int_1^\infty \frac{1}{(1+s)^{1/4}}\frac{1}{1+\exp\left(\frac{\mu s}{T}\right)}\,\mathrm{d}s \\
&\le C\mu^{3/4}
\end{aligned}$$

since $T \le C\mu$. The integral $\int_{\mathbb{R}^3\times\mathbb{R}^3} |V(x)||V(y)||x-y|\,\mathrm{d}x\,\mathrm{d}y < \infty$ is finite by the assumptions on V. Thus $\|A_{T,\mu}\|_2 \le C\mu^{3/4} \to 0$ vanishes as desired. □

Thus for our analysis, we may restrict to T's satisfying $T = o(\mu)$. By lemma 4.8 we thus have, in particular, that $\tilde{m}_\mu(T) \gg \sqrt{\mu}$ for such T's.

Proposition 4.13. *We have*

$$\lim_{\mu\to 0}\sup_{T=o(\mu)} \frac{\|A_{T,\mu}\|_2}{\mu^{1/4}\tilde{m}_\mu(T)} = 0$$

Proof. We use a similar but more refined bound as above on the kernel of $A_{T,\mu}$. Let $Z > 0$ be arbitrary. First,

$$\begin{aligned}
&\left|\frac{\sin|p||x-y|}{|p||x-y|}\right| \\
&\qquad \le C\left[p^2 Z^2 1_{\{|x-y|\le Z\}} + |p|^{1/2}|x-y|^{1/2}1_{\{|x-y|\ge Z\}}\right]1_{\{p^2\le 2\mu\}} + C|p|^{1/2}|x-y|^{1/2}1_{\{p^2\ge 2\mu\}}
\end{aligned}$$

where the constants don't depend on Z. Thus we bound

$$|A_{T,\mu}(x,y)| \le C|V(x)|^{1/2}|V(y)|^{1/2}\left[Z^2\int_0^{\sqrt{2\mu}}\left|\frac{1}{K_T}-\frac{1}{p^2}\right|p^4\,\mathrm{d}p\right.$$
$$\left.+|x-y|^{1/2}1_{\{|x-y|>Z\}}\int_0^{\sqrt{2\mu}}\left|\frac{1}{K_T}-\frac{1}{p^2}\right|p^{5/2}\,\mathrm{d}p+|x-y|^{1/2}\int_{\sqrt{2\mu}}^{\infty}\left|\frac{1}{K_T}-\frac{1}{p^2}\right|p^{5/2}\,\mathrm{d}p\right].$$

Since we may bound

$$\int_0^{\sqrt{2\mu}}\left|\frac{1}{K_T}-\frac{1}{p^2}\right|p^\alpha\,\mathrm{d}p \le \int_0^{\sqrt{2\mu}}\left(\frac{1}{K_T}-\frac{1}{p^2}\right)p^\alpha+2p^{\alpha-2}\,\mathrm{d}p \le C\tilde{m}_\mu(T)\mu^{\alpha/2}+C\mu^{\frac{\alpha-1}{2}} \le C\tilde{m}_\mu(T)\mu^{\frac{\alpha-2}{2}}.$$

We may bound the first integral by (a constant times) $\mu\tilde{m}_\mu(T)$ and the second by $\mu^{1/4}\tilde{m}_\mu(T)$. Similarly as before, the latter integral is bounded by $\mu^{3/4}$. Putting this together we conclude that

$$\limsup_{\mu\to 0}\sup_{T=o(\mu)}\frac{\|A_{T,\mu}\|_2}{\mu^{1/4}\tilde{m}_\mu(T)} \le C\left(\int_{|x-y|>Z}|V(x)||V(y)||x-y|\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}.$$

Since $|V(x)||V(y)||x-y|$ is integrable and $Z>0$ was arbitrary, we conclude the desired. □

We now rewrite $1+B_T$. For simplicity define $Q := V^{1/2}\frac{1}{p^2}|V|^{1/2}$. Then

$$1+B_T = (1+Q)\left(1+\frac{\tilde{m}_\mu(T)}{1+Q}\left(\left|V^{1/2}\right\rangle\left\langle|V|^{1/2}\right|+\frac{A_{T,\mu}}{\tilde{m}_\mu(T)}\right)\right).$$

The critical temperature is the largest temperature T such that B_T has -1 as an eigenvalue. Thus T_c is the largest temperature T such that

$$\frac{\tilde{m}_\mu(T)}{1+Q}\left(\left|V^{1/2}\right\rangle\left\langle|V|^{1/2}\right|+\frac{A_{T,\mu}}{\tilde{m}_\mu(T)}\right)$$

has -1 as an eigenvalue. Now, $\frac{1}{1+Q}\left|V^{1/2}\right\rangle\left\langle|V|^{1/2}\right|$ has $4\pi a$ as its only non-zero eigenvalue (it is rank one), and the lemma above gives that $\frac{A_{T,\mu}}{\tilde{m}_\mu(T)}$ vanishes as $\mu\to 0$. Also $\tilde{m}_\mu(T)$ is decreasing in T. Hence

$$\lim_{\mu\to 0}\tilde{m}_\mu(T_c) = \frac{-1}{4\pi a}.$$

We now show that this convergence is $o(\mu^{1/2})$.

Since $\tilde{m}_\mu(T)$ diverges in $T=0$ we get that $T_c>0$ for μ sufficiently small. Also we see that $\tilde{m}_\mu(T_c)$ is of order 1 in the limit $\mu\to 0$. Hence $A_{T,\mu}$ vanishes for $\mu\to 0$ and so $1+Q+A_{T,\mu}$ is invertible for μ small. Rewrite now instead

$$1+B_T = \left(1+Q+A_{T,\mu}\right)\left(1+\frac{\tilde{m}_\mu(T)}{1+Q+A_{T,\mu}}\left|V^{1/2}\right\rangle\left\langle|V|^{1/2}\right|\right)$$

Thus, T_c is characterised by the right-most operator having an eigenvalue of -1. Since this operator is rank one, this means

$$-\frac{1}{\tilde{m}_\mu(T_c)} = \left\langle|V|^{1/2}\middle|\frac{1}{1+Q+A_{T_c,\mu}}\middle|V^{1/2}\right\rangle. \tag{4.1}$$

We now consider the middle operator. First, note that

$$1 + Q + A_{T_c,\mu} = \left(1 + A_{T_c,\mu}\frac{1}{1+Q}\right)(1+Q).$$

Thus, by a power series expansion

$$\begin{aligned}
\frac{1}{1+Q+A_{T_c,\mu}} &= \frac{1}{1+Q}\left(\sum_{k=0}^{\infty}\left(-A_{T_c,\mu}\frac{1}{1+Q}\right)^k\right) \\
&= \frac{1}{1+Q} - \frac{1}{1+Q}A_{T_c,\mu}\frac{1}{1+Q} + \frac{1}{1+Q}A_{T_c,\mu}\left[\frac{1}{1+Q}\sum_{k=0}^{\infty}\left(-A_{T_c,\mu}\frac{1}{1+Q}\right)^k\right]A_{T_c,\mu}\frac{1}{1+Q} \\
&= \frac{1}{1+Q} - \frac{1}{1+Q}A_{T_c,\mu}\frac{1}{1+Q} + \frac{1}{1+Q}A_{T_c,\mu}\frac{1}{1+Q+A_{T_c,\mu}}A_{T_c,\mu}\frac{1}{1+Q}.
\end{aligned}$$

Plugging this into equation (4.1) above, the first term gives $4\pi a$ and the third term is $o(\mu^{1/2})$ since $\tilde{m}_\mu(T_c)$ is of order 1 and so by proposition 4.13 we have $\left\|A_{T_c,\mu}\right\| = o(\mu^{1/4})$. The second term gives

$$\left\langle f\middle|\operatorname{sgn} V A_{T_c,\mu}\middle|f\right\rangle, \qquad \text{with} \quad f = \frac{1}{1+Q}V^{1/2}.$$

We now show that this terms is also $o(\mu^{1/2})$.

Proposition 4.14. *The function f satisfies $f(x)|V(x)|^{1/2}(1+|x|) \in L^1$.*

Proof. First, using $\frac{1}{1+Q} = 1 - Q\frac{1}{1+Q}$ we have

$$\begin{aligned}
f(x)|V(x)|^{1/2}(1+|x|) &= \left(|V|^{1/2}(1+|\cdot|)\frac{1}{1+Q}V^{1/2}\right)(x) \\
&= \left(|V|^{1/2}(1+|\cdot|)V^{1/2}\right)(x) - \left(|V|^{1/2}(1+|\cdot|)Q\frac{1}{1+Q}V^{1/2}\right)(x).
\end{aligned}$$

For the first factor, we may bound its L^1-norm by $\|V(x)(1+|x|)\|_{L^1}$. For the second we may bound its L^1-norm by

$$\begin{aligned}
&\left\||V|^{1/2}(1+|x|)^{1/2}\right\|_{L^2}\left\|(1+|x|)^{1/2}V^{1/2}\frac{1}{p^2}|V|^{1/2}\right\|\left\|\frac{1}{1+Q}\right\|\left\|V^{1/2}\right\|_{L^2} \\
&\qquad\leq C\,\|V(x)(1+|x|)\|_{L^1}^{1/2}\,\|V\|_{L^1}^{1/2}\left\|(1+|x|)^{1/2}V^{1/2}\frac{1}{p^2}|V|^{1/2}\right\|.
\end{aligned}$$

We now show that this latter norm is finite. Since $\frac{1}{p^2}$ has Fourier transform $\frac{1}{|x|}$ up to a constant we have for the square of the Hilbert-Schmidt norm

$$\begin{aligned}
\left\|(1+|x|)^{1/2}V^{1/2}\frac{1}{p^2}|V|^{1/2}\right\|_2^2 &= C\int_{\mathbb{R}^3\times\mathbb{R}^3}\left|(1+|x|)^{1/2}V(x)^{1/2}\frac{1}{|x-y|}|V(y)|^{1/2}\right|^2 \mathrm{d}x\,\mathrm{d}y \\
&= C\int_{\mathbb{R}^3\times\mathbb{R}^3}(1+|x|)|V(x)|\frac{1}{|x-y|^2}|V(y)|\,\mathrm{d}x\,\mathrm{d}y \\
&\leq C\,\|V\|_{L^{3/2}}\,\|(1+|x|)V(x)\|_{L^{3/2}} < \infty
\end{aligned}$$

by the Hardy-Littlewood-Sobolev inequality [19, Thm. 4.3]. We conclude the desired. □

Proposition 4.15. *In the limit $\mu \to 0$ we have* $\langle f | \operatorname{sgn} V A_{T_c,\mu} | f \rangle = o(\mu^{1/2})$.

Proof. First,

$$\begin{aligned}&\langle f | \operatorname{sgn} V A_{T_c,\mu} | f \rangle \\ &\quad = \frac{1}{2\pi^2} \int_{\mathbb{R}^3\times\mathbb{R}^3} \overline{f(x)} |V(x)|^{1/2} f(y) |V(y)|^{1/2} \int_0^\infty \left(\frac{\sin p|x-y|}{p|x-y|} - 1 \right) \left(\frac{1}{K_T} - \frac{1}{p^2} \right) p^2 \, \mathrm{d}p \, \mathrm{d}x \, \mathrm{d}y.\end{aligned}$$

Similarly as in the proof of proposition 4.13 we decompose the $\sin b/b - 1$ term. We get the following bound on the kernel of $A_{T_c,\mu}$.

$$\begin{aligned}\left| A_{T_c,\mu}(x,y) \right| \leq C |V(x)|^{1/2} |V(y)|^{1/2} &\left[Z^2 \int_0^{\sqrt{2\mu}} \left| \frac{1}{K_{T_c}} - \frac{1}{p^2} \right| p^4 \, \mathrm{d}p \right. \\ &\left. + |x-y| 1_{\{|x-y|>Z\}} \int_0^{\sqrt{2\mu}} \left| \frac{1}{K_{T_c}} - \frac{1}{p^2} \right| p^3 \, \mathrm{d}p + |x-y|^{1/2} \int_{\sqrt{2\mu}}^\infty \left| \frac{1}{K_{T_c}} - \frac{1}{p^2} \right| p^{5/2} \, \mathrm{d}p \right].\end{aligned}$$

These integrals are bounded by μ, $\mu^{1/2}$, and $\mu^{3/4}$ respectively (recall that $\tilde{m}_\mu(T_c)$ is of order 1) by the same arguments as in proposition 4.11. We thus get for the expectation of f

$$\begin{aligned}\left| \langle f | \operatorname{sgn} V A_{T_c,\mu} | f \rangle \right| \quad \leq \quad C \int & |f(x)| |V(x)|^{1/2} |f(y)| |V(y)|^{1/2} \\ & \times \left[Z^2 \mu + |x-y| 1_{\{|x-y|>Z\}} \sqrt{\mu} + |x-y|^{1/2} \mu^{3/4} \right] \, \mathrm{d}x \, \mathrm{d}y.\end{aligned}$$

Since Z was arbitrary, we may conclude the desired similarly as in the proof of proposition 4.13. □

Going back to equation (4.1) we thus see that $\tilde{m}_\mu(T_c) = \frac{-1}{4\pi a} + o(\mu^{1/2})$. With lemma 4.8 we thus have

$$\lim_{\mu \to 0} \left(\frac{1}{\sqrt{\mu}} \left(\frac{\mu}{2\pi^2} \frac{1}{\sqrt{\mu}} \left(\log \frac{\mu}{T_c} + \gamma - 2 + \log \frac{8}{\pi} \right) + \frac{1}{4\pi a} \right) \right) = 0.$$

That is,

$$\lim_{\mu \to 0} \left(\log \frac{\mu}{T_c} + \frac{\pi}{2\sqrt{\mu} a} \right) = 2 - \gamma - \log \frac{8}{\pi}.$$

This proves theorem 4.10.

5 The Energy Gap

The function Δ is, as stated before, related to an energy gap for the superconductor. This we will discuss in this section. By introducing an approximate Hamiltonian as in [16] for the temperature $T = 0$ one sees that $E_\Delta(p) = \sqrt{(p^2 - \mu)^2 + |\Delta(p)|^2}$ has the interpretation as the dispersion relation of certain quasi-particles. In a sense such a quasi-particle is a mixture of a hole and an electron. This means that the minimum $\Xi := \inf E_\Delta(p)$ has the interpretation as an energy gap of the system. See [2, p. 270-276] for a discussion of these quasi-particles and the dispersion relation. We now study this energy gap Ξ.

5.1 Weak Coupling

We study the asymptotics of the energy gap Ξ in the limit of weak coupling, meaning again that we consider a potential λV for V fixed and $\lambda > 0$ some small number. We will show that the energy gap Ξ is exponentially small in the coupling. The techniques used in this section are much the same as in section 4.2. This section is also based on [13].

First, for the temperature $T = 0$ we saw that $\Gamma(p) = \Gamma(-p)$ was a projection and that for a given α the minimising γ would be maximal for $p^2 < \mu$ and minimal for $p^2 > \mu$. This means

$$\gamma(p) = \begin{cases} \frac{1}{2}\left(1 + \sqrt{1 - 4|\hat{\alpha}(p)|^2}\right) & \text{if } p^2 < \mu \\ \frac{1}{2}\left(1 - \sqrt{1 - 4|\hat{\alpha}(p)|^2}\right) & \text{if } p^2 > \mu \end{cases}.$$

Thus by subtracting a constant we get the BCS functional for zero temperature

$$\mathcal{F}_0(\alpha) = \frac{1}{2}\int |p^2 - \mu|\left(1 - \sqrt{1 - 4|\hat{\alpha}(p)|^2}\right) \mathrm{d}p + \lambda \int V(x)|\alpha(x)|^2 \,\mathrm{d}x.$$

For radial potentials V, this functional is invariant under rotations, i.e. $\alpha(x) \mapsto \alpha(Rx)$ for any rotation $R \in SO(3)$. Hence if $\alpha(x)$ is a minimiser, so is $\alpha(Rx)$. We prove the following theorem.

Theorem 5.1. *Let $V \in L^1 \cap L^{3/2}$ be radial with $\hat{V} \leq 0$ and $\hat{V}(0) < 0$, and let $\mu > 0$. Then there exists a unique (up to a constant global phase) minimiser α of the BCS functional at temperature $T = 0$. The associated energy gap Ξ is strictly positive and*

$$\lim_{\lambda \to 0}\left(\log \frac{\mu}{\Xi} + \frac{\pi}{2\sqrt{\mu} b_\mu(\lambda)}\right) = 2 - \log 8$$

where $b_\mu(\lambda)$ is as defined in section 4.2.

That is, for weak coupling the energy gap is

$$\Xi = \mu\left(8e^{-2} + o(1)\right) \exp\left(\frac{\pi}{2\sqrt{\mu} b_\mu(\lambda)}\right).$$

Together with theorem 4.7 we immediately have the following.

Corollary 5.2. *Let $V \in L^1 \cap L^{3/2}$ be radial with $\hat{V} \leq 0$ and $\hat{V}(0) < 0$, and let $\mu > 0$. Then the energy gap Ξ and critical temperature T_c satisfies*

$$\lim_{\lambda \to 0} \frac{\Xi}{T_c} = \pi e^{-\gamma} \approx 1.7639.$$

Note that since $\int V \,\mathrm{d}x = (2\pi)^{3/2}\hat{V}(0) < 0$ for such V's we have that $e_\mu < 0$ and so the assumptions of theorem 4.7 are satisfied. Notice also that the constant $\pi e^{-\gamma}$ is independent on the potential V and the chemical potential μ. Thus, the ratio of the energy gap and critical temperature tends to some universal constant in this limit. Such a universal ratio has been observed before in the physics literature, [4, 21].

Now, to prove theorem 5.1. First, we recall some facts from section 3. The Euler-Lagrange equations are

$$-\Delta(p) = 2E_\Delta(p)\hat{\alpha}(p) \qquad \text{and} \qquad \gamma(p) = \frac{1}{2} - \frac{p^2 - \mu}{2E_\Delta(p)} = \begin{cases} \frac{1}{2}\left(1 + \sqrt{1 - 4|\hat{\alpha}(p)|^2}\right) & \text{if } p^2 < \mu \\ \frac{1}{2}\left(1 - \sqrt{1 - 4|\hat{\alpha}(p)|^2}\right) & \text{if } p^2 > \mu \end{cases},$$

where $E_\Delta(p) = \sqrt{(p^2-\mu)^2 + |\Delta(p)|^2} = K^\Delta_{T=0}(p)$ and $\Delta = 2\lambda\widehat{V\alpha}$. Using the above we get

$$E_\Delta(p) = \frac{|p^2-\mu|}{\sqrt{1-4|\hat\alpha(p)|^2}}.$$

Thus we have

$$-\Delta(p) = 2\frac{|p^2-\mu|}{\sqrt{1-4|\hat\alpha(p)|^2}}\hat\alpha(p). \tag{5.1}$$

Of course Δ also satisfies the BCS gap equation for Δ,

$$\Delta(p) = -\frac{\lambda}{(2\pi)^{3/2}}\int \hat V(p-q)\frac{\Delta(q)}{E_\Delta(q)}\,\mathrm{d}q.$$

Lemma 5.3. *There exists a unique minimiser α with $\hat\alpha > 0$. Moreover, any minimiser of $\mathcal{F}_0$ is unique up to a constant global phase.*

The proof shows that $-\Delta$ is strictly positive (for this choice of minimiser α). Also, the BCS gap equation for Δ gives that Δ is continuous and so $\Xi > 0$ is strictly positive.

Proof. Since $\int V\,\mathrm{d}x = (2\pi)^{3/2}\hat V(0) < 0$ we have that $e_\mu < 0$ and so by theorem 4.3 we have that $T_c > 0$. Thus, a minimiser is non-zero. Let $\alpha \not\equiv 0$ be such a minimiser. Then

$$(2\pi)^{3/2}\langle\alpha|V|\alpha\rangle = \iint \hat\alpha(p)\hat V(p-q)\hat\alpha(q)\,\mathrm{d}q\,\mathrm{d}p \geq \iint |\hat\alpha(p)|\hat V(p-q)|\hat\alpha(q)|\,\mathrm{d}q\,\mathrm{d}p \tag{5.2}$$

since $\hat V \leq 0$. Thus (the inverse Fourier transform of) $|\hat\alpha|$ is also a minimiser of $\mathcal{F}_0$.

Let now for contradiction $f \neq g$ be two minimisers with different non-negative Fourier transform. Then by strict convexity of $t \mapsto 1-\sqrt{1-4t^2}$ the function $\psi := \frac{1}{\sqrt2}f + \frac{i}{\sqrt2}g$ satisfies

$$\mathcal{F}_0(\psi) < \frac12\mathcal{F}_0(f) + \frac12\mathcal{F}_0(g).$$

Contradiction. Thus there exists a unique minimiser with non-negative Fourier transform. Let now, α be this unique minimiser with non-negative Fourier transform. We show that $\hat\alpha$ is non-vanishing, i.e. $\hat\alpha > 0$. Since $\hat\alpha \geq 0$ we have by equation (5.1) that $-\Delta \geq 0$.

The set $\{\Delta = 0\}$ is closed by the continuity of Δ. We show, that it is also open. Thus, suppose that $\Delta(p) = 0$. Since $\hat V(0) < 0$ we have $\hat V < -\varepsilon$ on some ball $B(0,r)$. Thus by the BCS gap equation for Δ we have $\Delta \equiv 0$ on $B(p,r)$, that is $\{\Delta = 0\}$ is open. By connectedness we have either $\Delta \equiv 0$ or Δ non-vanishing. Since $\alpha \not\equiv 0$ we are in the latter case and so $-\Delta > 0$ everywhere. Hence $\hat\alpha > 0$ everywhere.

Now, let α be any minimiser. By the above we see that $|\hat\alpha(p)|$ is non-vanishing, i.e. $\hat\alpha$ is non-vanishing. Equation (5.2) above is strict for non-vanishing $\hat\alpha$, unless $\hat\alpha(p) = e^{i\phi}|\hat\alpha(p)|$ for some constant $\phi \in \mathbb{R}$. Hence our α satisfies this. This shows the desired uniqueness. □

Remark 5.4. We can use this to prove that a minimiser must be rotationally symmetric as follows. Let α be any minimiser. Then $|\hat\alpha|$ is unique and still (the Fourier transform of) a minimiser. Any rotation is also a minimiser so by uniqueness we have $|\hat\alpha(Rp)| = |\hat\alpha(p)|$ for any rotation R. Thus $\hat\alpha(p) = e^{i\phi}|\hat\alpha(p)| = e^{i\phi}|\hat\alpha(Rp)| = \hat\alpha(Rp)$ is rotationally symmetric.

In the following we will always take α to be the unique minimiser with positive Fourier transform.

The BCS gap equation for α is $(E_\Delta + V)\alpha = 0$. In fact,

Lemma 5.5. *The minimising α above is the ground state of the pseudo-differential operator $E_\Delta + V$.*

In particular $E_\Delta + V \geq 0$.

Proof. Since $\hat{V} \leq 0$ we may take the ground state ψ to have $\hat{\psi} \geq 0$ by a computation similar to equation (5.2). Then $\langle \psi | \alpha \rangle = \left\langle \hat{\psi} \middle| \hat{\alpha} \right\rangle \neq 0$ so they are not orthogonal, and so α is the ground state. □

We may as before use the Birman-Schwinger principle to see that the operator

$$B_\Delta := \lambda V^{1/2} \frac{1}{E_\Delta} |V|^{1/2}$$

has -1 as its lowest eigenvalue. The eigenvector is $\phi = V^{1/2}\alpha$. We decompose this operator as

$$\lambda V^{1/2} \frac{1}{E_\Delta} |V|^{1/2} = \lambda m_\mu(\Delta) V^{1/2} \mathfrak{F}^\dagger \mathfrak{F} |V|^{1/2} + \lambda V^{1/2} M_\Delta |V|^{1/2},$$

where

$$m_\mu(\Delta) = \frac{1}{4\pi\mu} \int \frac{1}{E_\Delta(p)} - \frac{1}{p^2} \,\mathrm{d}p.$$

Similarly as in lemma 4.5 we have

Proposition 5.6. *The operator $V^{1/2} M_\Delta |V|^{1/2}$ is Hilbert-Schmidt. Moreover, its Hilbert-Schmidt norm is bounded uniformly in small λ.*

This is not proven in [13]. There it is noted that an argument similar to that in the proof of lemma 4.5 works. This we have not been able to see. Instead, we give our own proof. Some of the bounds in the proof are however similar to those in the proof of lemma 4.5.

Proof. First we decompose

$$M_\Delta = \frac{1}{E_\Delta} 1_{\{p^2 < 2\mu\}} + \frac{1}{E_\Delta} 1_{\{p^2 \geq 2\mu\}} - m_\mu(\Delta) \mathfrak{F}^\dagger \mathfrak{F}.$$

The second term can be bounded by (a constant times) $\frac{1}{p^2}$. So the kernel of $V^{1/2} \frac{1}{E_\Delta} 1_{\{p^2 \geq 2\mu\}} |V|^{1/2}$ is bounded by $|V(x)|^{1/2} \frac{1}{|x-y|} |V(y)|^{1/2} \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ by the Hardy-Littlewood-Sobolev inequality [19, Thm. 4.3] and the fact that the Fourier transform of $\frac{1}{p^2}$ is $\frac{1}{|x|}$. The remaining operator has the kernel

$$\begin{aligned}
\left(M_\Delta - \frac{1}{E_\Delta} 1_{\{p^2 \geq 2\mu\}} \right)(x,y) &= \frac{1}{(2\pi)^3} \int_{\{p^2 < 2\mu\}} \frac{e^{ip(x-y)}}{E_\Delta} \,\mathrm{d}p - \frac{1}{(2\pi)^3} \frac{\sin \sqrt{\mu}|x-y|}{\sqrt{\mu}|x-y|} \int \frac{1}{E_\Delta} - \frac{1}{p^2} \,\mathrm{d}p \\
&= \frac{1}{2\pi^2} \left[\int_0^{\sqrt{2\mu}} \frac{k^2}{E_\Delta(k)} \frac{\sin k|x-y|}{k|x-y|} - \frac{\sin \sqrt{\mu}|x-y|}{\sqrt{\mu}|x-y|} \,\mathrm{d}k \right. \\
&\quad + \sqrt{2\mu} \frac{\sin \sqrt{\mu}|x-y|}{\sqrt{\mu}|x-y|} \\
&\quad \left. - \int_{\sqrt{2\mu}}^\infty \left(\frac{k^2}{E_\Delta(k)} - 1 \right) \frac{\sin \sqrt{\mu}|x-y|}{\sqrt{\mu}|x-y|} \,\mathrm{d}k \right],
\end{aligned}$$

where we again used that $\int_{S^2} e^{ipx} \,\mathrm{d}\omega(p) = 4\pi \frac{\sin|x|}{|x|}$ and that $\mathfrak{F}^\dagger \mathfrak{F}$ has kernel $\frac{\sqrt{\mu}}{2\pi^2} \frac{\sin \sqrt{\mu}|x-y|}{|x-y|}$. Here $\Delta(k)$ means the values of $\Delta(p)$ on the sphere with $|p| = k$. By the radial symmetry of V, this is well-defined. Define operators $M_\Delta^{(i)}, i = 1,2,3$ by having the kernels on the right hand side.

The second term, $M_\Delta^{(2)}$, is again taken care of by the Hardy-Littlewood-Sobolev inequality. For the first term, $M_\Delta^{(1)}$ we use the bound $\left| \frac{\sin a}{a} - \frac{\sin b}{b} \right| \leq C \frac{|a-b|}{a+b}$ valid for any $a, b > 0$. Then

$$\left| \frac{k^2}{E_\Delta(k)} \frac{\sin k|x-y|}{k|x-y|} - \frac{\sin \sqrt{\mu}|x-y|}{\sqrt{\mu}|x-y|} \right| \leq C \frac{|k - \sqrt{\mu}|}{k + \sqrt{\mu}} \leq C.$$

This is integrable so $M_\Delta^{(1)}$ has bounded kernel. Thus $V^{1/2}(x)M_\Delta^{(1)}(x,y)|V^{1/2}(y)| \in L^2(\mathbb{R}^3\times\mathbb{R}^3)$ as desired.

For the third term, $M_\Delta^{(3)}$, we compute the integral

$$\left|\int_{\sqrt{2\mu}}^\infty \frac{k^2}{E_\Delta(k)} - 1\,\mathrm{d}k\right| = \left|\int_{\sqrt{2\mu}}^\infty \frac{2\mu k^2 - \mu^2 - |\Delta(k)|^2}{E_\Delta(k)\,(k^2+E_\Delta(k))}\,\mathrm{d}k\right| \le C\int_{\sqrt{2\mu}}^\infty \frac{2\mu k^2}{k^4} + \frac{|\Delta(k)|^2}{k^4}\,\mathrm{d}k.$$

The first term here is integrable. The latter may be bounded by $\|\Delta\|_{L^\infty}^2$. This we bound as follows.

$$\|\Delta\|_{L^\infty} = 2\lambda\left\|\widehat{V\alpha}\right\|_{L^\infty} \le C\lambda\,\|V\alpha\|_{L^1} \le C\lambda\,\|V\|_{L^{3/2}}\,\|\alpha\|_{L^3} \le C\lambda\,\|V\|_{L^{3/2}}\,\|\alpha\|_{L^2}^{1/2}\,\|\alpha\|_{L^6}^{1/2}$$

by [8, Prop. 6.10]. The L^6-norm is bounded by the H^1-norm by Sobolev's inequality [19, Thm. 8.3]. Also, the H^1-norm of α is bounded uniformly in small λ by the proof of theorem 3.1. (The constant A in that proof (can be slightly changed to one that) is increasing in λ and the minimiser satisfies $\|\alpha\|_{H^1} \le 8A$.) We conclude that $M_\Delta^{(3)}(x,y)$ is uniformly bounded in small λ and so we conclude the desired. □

Factoring

$$1 + B_\Delta = \left(1 + \lambda V^{1/2}M_\Delta|V|^{1/2}\right)\left(1 + \frac{\lambda m_\mu(\Delta)}{1+\lambda V^{1/2}M_\Delta|V|^{1/2}}V^{1/2}\mathfrak{F}^\dagger\mathfrak{F}|V|^{1/2}\right)$$

as in section 4.2 we see that the operator

$$T_\Delta := \mathfrak{F}|V|^{1/2}\frac{\lambda m_\mu(\Delta)}{1+\lambda V^{1/2}M_\Delta|V|^{1/2}}V^{1/2}\mathfrak{F}^\dagger$$

has -1 as its lowest eigenvalue. Inverting the middle operator by a Neumann series and recalling $\mathfrak{F}V\mathfrak{F}^\dagger = \sqrt{\mu}\mathcal{V}_\mu$ we thus get

$$-1 = \inf\operatorname{spec}\left(\lambda m_\mu(\Delta)\left(\sqrt{\mu}\mathcal{V}_\mu + \lambda\mathfrak{F}VM_\Delta V\mathfrak{F}^\dagger + O(\lambda^2)\right)\right).$$

That is,

$$\lim_{\lambda\to0}\left(m_\mu(\Delta) + \frac{1}{\inf\operatorname{spec}\left(\lambda\sqrt{\mu}\mathcal{V}_\mu + \lambda^2\mathfrak{F}VM_\Delta V\mathfrak{F}^\dagger\right)}\right) = 0 \tag{5.3}$$

In particular $\lambda m_\mu(\Delta) \to \frac{-1}{\sqrt{\mu}e_\mu}$.

Equation (5.3) is what we will use to prove theorem 5.1. We will now study more precisely the asymptotics of $m_\mu(\Delta)$ and $\mathfrak{F}VM_\Delta V\mathfrak{F}^\dagger$.

Lemma 5.7. *For small λ the function Δ satisfies*

$$\Delta(p) = f(\lambda)\left(\int_{\Omega_\mu}\hat{V}(p-q)\,\mathrm{d}\omega(q) + \lambda\eta_\lambda(p)\right)$$

for some $f(\lambda)$ and function η_λ with $\|\eta_\lambda\|_{L^\infty} \le C$ bounded independently of λ.

Proof. Since ϕ is an eigenvector for B_Δ we see that $\mathfrak{F}|V|^{1/2}\phi$ is an eigenvector of T_Δ with eigenvalue -1. Also $u \equiv \frac{1}{\sqrt{4\pi\mu}}$ is an eigenvector of T_Δ, since T_Δ is rotationally symmetric. Now, u is the unique eigenvector of $\mathcal{V}_\mu$ with its lowest eigenvalue e_μ as discussed in remark 4.4. Thus, for small enough

λ, we have that u must be the (unique) eigenvector for the smallest eigenvalue of T_Δ. Hence, by transferring u to a (thus unique) eigenvector for B_Δ, we have

$$\phi = f(\lambda)\frac{1}{1+\lambda V^{1/2}M_\Delta|V|^{1/2}}V^{1/2}\mathfrak{F}^\dagger u = f(\lambda)\left(V^{1/2}\mathfrak{F}^\dagger u + \lambda\xi_\lambda\right) \tag{5.4}$$

for some $f(\lambda)$ and by a Neumann series. The vector ξ_λ satisfies $\|\xi_\lambda\|_{L^2} \le C$ uniformly in small λ since $V^{1/2}M_\Delta|V|^{1/2}$ and $V^{1/2}\mathfrak{F}^\dagger$ are bounded uniformly in λ. Thus

$$\Delta = 2\lambda\widehat{V\alpha} = 2\lambda\widehat{|V|^{1/2}\phi} = 2\lambda f(\lambda)\left(\widehat{V\mathfrak{F}^\dagger u} + \lambda\eta_\lambda\right).$$

where $\eta_\lambda = \widehat{|V|^{1/2}\xi_\lambda}$. Now we compute these two terms.

$$\|\eta_\lambda\|_{L^\infty} \le \frac{1}{(2\pi)^{3/2}}\left\||V|^{1/2}\xi_\lambda\right\|_{L^1} \le \frac{1}{(2\pi)^{3/2}}\|V\|_{L^1}^{1/2}\|\xi_\lambda\|_{L^2} \le C$$

uniformly in λ. Also, one computes that

$$\widehat{V\mathfrak{F}^\dagger u}(p) = \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{4\pi\mu}}\int_{\Omega_\mu}\hat{V}(p-q)\,\mathrm{d}\omega(q).$$

Thus, absorbing the constant and the pre-factor of λ into $f(\lambda)$ and rescaling η_λ we get the desired. □

In the limit $\lambda \to 0$ we have $\Delta \to 0$ pointwise at least, and thus $f(\lambda) \to 0$. Moreover, $f(\lambda)$ is continuous in λ. This is seen from the defining equation for $f(\lambda)$, (the leftmost equality in equation (5.4)). The functions on either side of the equality are continuous in λ.

We are now ready to prove the following analogue of lemma 4.8

Lemma 5.8. *In the limit $\lambda \to 0$ we have*

$$m_\mu(\Delta) = \frac{1}{4\pi\mu}\int \frac{1}{\sqrt{(p^2-\mu)^2+|\Delta(p)|^2}} - \frac{1}{p^2}\,\mathrm{d}p = \frac{1}{\sqrt{\mu}}\left(\log\frac{\mu}{-\Delta(\sqrt{\mu})} - 2 + \log 8 + o(1)\right).$$

Here by $\Delta(\sqrt{\mu})$ we mean the value of Δ on the Fermi-sphere, i.e. $\Delta(p)$ for any p with $p^2 = \mu$. Since Δ is radial this is well-defined. The proof of similar to that of lemma 4.8.

Proof. First, by computing the spherical integral and a change of variables we have

$$\begin{aligned} m_\mu(\Delta) = \frac{1}{2\mu}\Bigg[\int_0^\mu &\frac{\sqrt{\mu-t}-\sqrt{\mu}}{\sqrt{t^2+\Delta(\sqrt{\mu-t})^2}} + \frac{\sqrt{\mu+t}-\sqrt{\mu}}{\sqrt{t^2+\Delta(\sqrt{\mu+t})^2}} - \frac{1}{\sqrt{\mu-t}} - \frac{1}{\sqrt{\mu+t}}\,\mathrm{d}t \\ + \int_0^\mu \frac{\sqrt{\mu}}{\sqrt{t^2+\Delta(\sqrt{\mu-t})^2}} &+ \frac{\sqrt{\mu}}{\sqrt{t^2+\Delta(\sqrt{\mu+t})^2}}\,\mathrm{d}t + \int_\mu^\infty \frac{\sqrt{\mu+t}}{\sqrt{t^2+\Delta(\sqrt{\mu+t})^2}} - \frac{1}{\sqrt{\mu+t}}\,\mathrm{d}t\Bigg]. \end{aligned} \tag{5.5}$$

Now, we claim that $\int_{\Omega_\mu}\hat{V}(p-q)\,\mathrm{d}\omega(q)$ is a Lipschitz continuous function of p. Indeed,

$$\left|\int_{\Omega_\mu}\hat{V}(p-q)-\hat{V}(r-q)\,\mathrm{d}\omega(q)\right| = \sqrt{\frac{2\mu}{\pi}}\left|\int V(x)\frac{\sin\sqrt{\mu}|x|}{|x|}\left(e^{-ipx}-e^{-irx}\right)\mathrm{d}x\right| \le C|p-r|.$$

We now claim that

$$m_\mu(\Delta) = \frac{1}{2\mu}\left[\int_0^\mu \frac{\sqrt{\mu-t}-\sqrt{\mu}}{t} + \frac{\sqrt{\mu+t}-\sqrt{\mu}}{t} - \frac{1}{\sqrt{\mu-t}} - \frac{1}{\sqrt{\mu+t}}\,\mathrm{d}t \right.$$
$$\left. + \int_0^\mu \frac{2\sqrt{\mu}}{\sqrt{t^2+\Delta(\sqrt{\mu})^2}}\,\mathrm{d}t + \int_\mu^\infty \frac{\sqrt{\mu+t}}{t} - \frac{1}{\sqrt{\mu+t}}\,\mathrm{d}t + o(1)\right]. \tag{5.6}$$

For the first and last integrals this follows by dominated convergence. For the middle integral we have the following. Define the function(s) $x(t) := \Delta(\sqrt{\mu \pm t})$. Then we want to show that

$$\int_0^\mu \frac{1}{\sqrt{t^2+x(t)^2}} - \frac{1}{\sqrt{t^2+x(0)^2}}\,\mathrm{d}t \longrightarrow 0. \tag{5.7}$$

We first show that $x(t)$ and $x(0)$ are of the same size for small t. With lemma 5.7 above and the Lipschitz continuity of $\int_{\Omega_\mu} \hat{V}(p-q)\,\mathrm{d}\omega(q)$ we have that

$$|x(t)-x(0)| \le |f(\lambda)|(Ct+C\lambda) \qquad \text{and} \qquad x(0) = f(\lambda)\left(\int_{\Omega_\mu} \hat{V}(\sqrt{\mu}-q)\,\mathrm{d}\omega(q) + O(\lambda)\right).$$

To make sense of $\sqrt{\mu}-q$ one need to choose any direction $\hat{p} \in S^2$, then this means $\sqrt{\mu}-q = \sqrt{\mu}\hat{p}-q$. By the radial symmetry this does not depend on the direction $\hat{p}$.

It follows that $|f(\lambda)| \le C|x(0)|$. Thus

$$|x(t)| \le |f(\lambda)|\left(\int_{\Omega_\mu} |\hat{V}(\sqrt{\mu \pm t}-q)|\,\mathrm{d}q + O(\lambda)\right) \le |f(\lambda)|\left(4\pi\mu \left\|\hat{V}\right\|_{L^\infty} + O(\lambda)\right) \le C|x(0)|$$

For the bound $|x(0)| \le C|x(t)|$ we do the following. We have

$$x(t) = f(\lambda)\left(\int_{\Omega_\mu} \hat{V}(\sqrt{\mu \pm t}-q)\,\mathrm{d}\omega(q) + O(\lambda)\right) = f(\lambda)\left(\int_{\Omega_\mu+\sqrt{\mu\pm t}} \hat{V}(q)\,\mathrm{d}\omega(q) + O(\lambda)\right).$$

To make sense of $\sqrt{\mu \pm t}-q$ and $\Omega_\mu + \sqrt{\mu \pm t}$ we do same as for $\sqrt{\mu}-q$ above.

By the continuity of $\hat{V}$ and that $\hat{V}(0) < 0$ we have that $\hat{V} < \frac{\hat{V}(0)}{2}$ in some small ball $B(0,\varepsilon')$. For t small, say $t < \varepsilon$ then the set $\Omega_\mu + \sqrt{\mu \pm t}$ intersects $B(0,\varepsilon')$ in some set of measure at least c_1 for some constant $c_1 > 0$. Thus we have

$$|x(t)| \ge |f(\lambda)|\left(\frac{|\hat{V}(0)|}{2}c_1 + O(\lambda)\right) \ge c|x(0)|$$

for such $t < \varepsilon$. Note that ε is an absolute constant, that does not depend on λ. We now use this to bound the integrand in equation (5.7) above. For $t > \varepsilon$ we have

$$\begin{aligned}\left|\frac{1}{\sqrt{t^2+x(t)^2}} - \frac{1}{\sqrt{t^2+x(0)^2}}\right| &= \frac{\left|x(t)^2-x(0)^2\right|}{\sqrt{t^2+x(t)^2}\sqrt{t^2+x(0)^2}\left(\sqrt{t^2+x(t)^2}+\sqrt{t^2+x(0)^2}\right)} \\ &\le \frac{|x(t)-x(0)|}{\sqrt{t^2+x(t)^2}\sqrt{t^2+x(0)^2}} \\ &\le C|x(0)|t^{-2}\end{aligned}$$

which clearly has vanishing integral in the limit $\lambda \to 0$. For $t < \varepsilon$ we have

$$\begin{aligned}\left|\frac{1}{\sqrt{t^2+x(t)^2}} - \frac{1}{\sqrt{t^2+x(0)^2}}\right| &= \frac{\left|x(t)^2 - x(0)^2\right|}{\sqrt{t^2+x(t)^2}\sqrt{t^2+x(0)^2}\left(\sqrt{t^2+x(t)^2}+\sqrt{t^2+x(0)^2}\right)} \\ &\leq C\frac{(t+\lambda)x(0)^2}{\sqrt{t^2+x(t)^2}\sqrt{t^2+x(0)^2}\left(t+\sqrt{t^2+x(0)^2}\right)} \\ &\leq C\frac{x(0)^2}{\sqrt{t^2+x(0)^2}\left(t+\sqrt{t^2+x(0)^2}\right)} + C\lambda\frac{|x(0)|}{\sqrt{t^2+x(0)^2}\left(t+\sqrt{t^2+x(0)^2}\right)}.\end{aligned}$$

Now, one may compute that

$$\int_0^\varepsilon \frac{|x(0)|}{\sqrt{t^2+x(0)^2}\left(t+\sqrt{t^2+x(0)^2}\right)}\,\mathrm{d}t = O(1).$$

Thus both terms in the bound above have vanishing integral in the limit $\lambda \to 0$. This proves equation (5.7) which in turn proves equation (5.6).

Now, it is a matter of computing the expression for $m_\mu(\Delta)$ in equation (5.6). The first and last integrals are computed in lemma 4.8. The middle gives

$$\int_0^\mu \frac{\sqrt{\mu}}{\sqrt{t^2+\Delta(\sqrt{\mu})^2}}\,\mathrm{d}t = \sqrt{\mu}\log\frac{\mu+\sqrt{\mu^2+\Delta(\sqrt{\mu})^2}}{-\Delta(\sqrt{\mu})} = \sqrt{\mu}\log\frac{2\mu}{-\Delta(\sqrt{\mu})} + o(1)$$

by hyperbolic substitutions. We conclude the desired. □

Similarly as in section 4.2 we also need to study the behaviour of $\mathfrak{F}VM_\Delta V\mathfrak{F}^\dagger$ in the limit $\lambda \to 0$. We have

$$\left\langle u\middle|\mathfrak{F}VM_\Delta V\mathfrak{F}^\dagger\middle|u\right\rangle = \int \frac{1}{E_\Delta}\left(|\psi(p)|^2 - |\psi(\sqrt{\mu}\hat{p})|^2\right) + \frac{1}{p^2}|\psi(\sqrt{\mu}\hat{p})|^2\,\mathrm{d}p$$

where

$$\psi(p) = \widehat{V\mathfrak{F}^\dagger u}(p) = \frac{1}{(2\pi)^{3/2}}\int_{\Omega_\mu}\hat{V}(p-q)u(q)\,\mathrm{d}\omega(q) = \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{4\pi\mu}}\int_{\Omega_\mu}\hat{V}(p-q)\,\mathrm{d}\omega(q).$$

Again, E_Δ decreases to $|p^2-\mu|$ and so by dominated convergence $\left\langle u\middle|\mathfrak{F}VM_\Delta V\mathfrak{F}^\dagger\middle|u\right\rangle$ converges to $\left\langle u\middle|\mathcal{W}_\mu\middle|u\right\rangle$.

Plugging all this into equation (5.3) we thus have

$$\lim_{\lambda\to 0}\left(\frac{1}{\sqrt{\mu}}\left(\log\frac{\mu}{-\Delta(\sqrt{\mu})} - 2 + \log 8\right) - \frac{1}{\left\langle u\middle|\lambda\sqrt{\mu}\mathcal{V}_\mu - \lambda^2\mathcal{W}_\mu\middle|u\right\rangle}\right) = 0.$$

Recalling that $b_\mu(\lambda) = \inf\operatorname{spec}\left(\frac{\pi}{2\sqrt{\mu}}\lambda\mathcal{V}_\mu - \frac{\pi}{2\mu}\lambda^2\mathcal{W}_\mu\right)$ we thus have (again with a similar argument as in section 4.2 to replace the expectation against u with the lowest eigenvalue)

$$\lim_{\lambda\to 0}\left(\log\frac{\mu}{-\Delta(\sqrt{\mu})} - \frac{\pi}{2\sqrt{\mu}b_\mu(\lambda)}\right) = 2 - \log 8.$$

We now want to replace $-\Delta(\sqrt{\mu})$ with the energy gap $\Xi = \inf E_\Delta(p)$. Since $E_\Delta(\sqrt{\mu}) = -\Delta(\sqrt{\mu})$ we clearly have $\Xi \leq -\Delta(\sqrt{\mu})$. Also

$$\Xi \geq \min_{|p^2-\mu|\leq\Xi} -\Delta(p) \geq -\Delta(\sqrt{\mu})(1+o(1))$$

since $\Xi \leq -\Delta(\sqrt{\mu}) = o(1)$ and so $\Xi = -\Delta(\sqrt{\mu})(1+o(1))$. We conclude that

$$\lim_{\lambda\to 0}\left(\log\frac{\mu}{\Xi} + \frac{\pi}{2\sqrt{\mu}b_\mu(\lambda)}\right) = 2 - \log 8.$$

That is, we have proven theorem 5.1.

5.2 Low Density

We study the asymptotics of the energy gap in the limit of low density. The results of this section are new and the result of my own work. Some of the methods used are very similar to (and inspired by) those of [14] presented in section 4.3. I expect to pursue publication of the contents of this section.

The main result we prove is the following.

Theorem 5.9. *Let V be radial and assume that $V(x)(1+|x|) \in L^1 \cap L^{3/2}$, $\hat{V} \leq 0$, $\hat{V}(0) < 0$, that $\|V\|_{L^{3/2}} < S_3$, and that the scattering length $a(V) < 0$. Then,*

$$\lim_{\mu\to 0}\left(\log\frac{\mu}{\Xi} + \frac{\pi}{2\sqrt{\mu}a}\right) = 2 - \log 8.$$

That is, in the limit of low density, the energy gap satisfies

$$\Xi = \mu\left(8e^{-2} + o(1)\right)\exp\left(\frac{\pi}{2\sqrt{\mu}a}\right).$$

This is known in the physics literature [18]. Recall that, the scattering length is

$$a(V) = \frac{1}{4\pi}\left\langle |V|^{1/2}\middle|\frac{1}{1+V^{1/2}\frac{1}{p^2}|V|^{1/2}}\middle|V^{1/2}\right\rangle.$$

Here $S_3 = \frac{3}{4}2^{2/3}\pi^{4/3} \approx 5.4779$ is the best constant in Sobolev's inequality [19, Thm. 8.3]. The assumption that $\|V\|_{L^{3/2}} < S_3$ gives that $p^2 + \lambda V > 0$ for any $\lambda \leq 1$ by [19, sect. 11.3]. Thus, by the Birman-Schwinger principle, the operator $\lambda V^{1/2}\frac{1}{p^2}|V|^{1/2}$ does not have -1 as an eigenvalue. Varying λ we thus get that the spectrum of $V^{1/2}\frac{1}{p^2}|V|^{1/2}$ is contained in $(-1,\infty)$. Thus, the scattering length is indeed finite. Also, for a V satisfying the assumptions it also satisfies the assumptions of theorem 4.10. We thus immediately get following.

Corollary 5.10. *Let V be radial and assume that $V(x)(1+|x|) \in L^1 \cap L^{3/2}$, $\hat{V} \leq 0$, $\hat{V}(0) < 0$, that $\|V\|_{L^{3/2}} < S_3$, and that the scattering length $a(V) < 0$. Then,*

$$\lim_{\mu\to 0}\frac{\Xi}{T_c} = \pi e^{-\gamma} \approx 1.7639.$$

This is the same universal ratio as in corollary 5.2. This universal ratio for low density is known in the physics literature, [11]. The rest of this section is dedicated to proving this theorem.

One of the key ideas in the proof is to study the asymptotics of

$$\tilde{m}_\mu(\Delta) = \frac{1}{(2\pi)^3}\int \frac{1}{E_\Delta(p)} - \frac{1}{p^2}\,\mathrm{d}p.$$

This is similar to what we did in sections 4.2, 4.3 and 5.1.

First, the BCS functional at zero temperature is

$$\mathcal{F}^{\mu,V}(\alpha) = \frac{1}{2}\int |p^2-\mu| \left(1-\sqrt{1-4|\hat{\alpha}(p)|^2}\right) \mathrm{d}p + \int V(x)|\alpha(x)|^2 \,\mathrm{d}x.$$

Since V satisfies the assumption of section 5.1 we have that there exists a unique minimiser with (strictly) positive Fourier transform. This we will denote by $\alpha_{\mu,V}$. By scaling we have that

$$\mathcal{F}^{\mu,V}(\alpha) = \mu^{5/2}\mathcal{F}^{1,\sqrt{\mu}V_{\sqrt{\mu}}}(\beta),$$

where $\beta(x) = \mu^{-3/2}\alpha(x/\sqrt{\mu})$ and $V_{\sqrt{\mu}}(x) = \mu^{-3/2}V(x/\sqrt{\mu})$. Note that $\left\|V_{\sqrt{\mu}}\right\|_{L^1} = \|V\|_{L^1}$ and $\left\|\sqrt{\mu}V_{\sqrt{\mu}}\right\|_{L^{3/2}} = \|V\|_{L^{3/2}}$. With this, we thus see that the minimiser with positive Fourier transform satisfies

$$\alpha_{\mu,V}(x) = \mu^{3/2}\alpha_{1,\sqrt{\mu}V_{\sqrt{\mu}}}(\sqrt{\mu}x).$$

We now bound this.

Proposition 5.11. *In the limit $\mu \to 0$ we have $\left\|\alpha_{\mu,V}\right\|_{H^1} \leq C\mu^{3/4}$.*

Proof. With the scaling argument above, we compute

$$\left\|\alpha_{\mu,V}\right\|_{H^1}^2 = \int |\hat{\alpha}_{\mu,V}(p)|^2\left(1+p^2\right) \mathrm{d}p = \mu^{3/2}\int |\hat{\alpha}_{1,\sqrt{\mu}V_{\sqrt{\mu}}}(q)|^2\left(1+\mu q^2\right) \mathrm{d}q \leq \mu^{3/2}\left\|\alpha_{1,\sqrt{\mu}V_{\sqrt{\mu}}}\right\|_{H^1}^2.$$

We now show, that this latter norm is bounded uniformly in μ.

Let $\lambda = \frac{S_3}{\|V\|_{L^{3/2}}} > 1$. Then, as $\left\|\sqrt{\mu}V_{\sqrt{\mu}}\right\|_{L^{3/2}} = \|V\|_{L^{3/2}}$ we have (as explained above) that $\frac{p^2}{\lambda} + \sqrt{\mu}V_{\sqrt{\mu}} \geq 0$. Thus we may bound for any α,

$$\begin{aligned}
\mathcal{F}^{1,\sqrt{\mu}V_{\sqrt{\mu}}}(\alpha) &\geq \int (p^2-1)|\hat{\alpha}(p)|^2\,\mathrm{d}p + \int \sqrt{\mu}V_{\sqrt{\mu}}(x)|\alpha(x)|^2\,\mathrm{d}x \\
&= \left\langle \alpha \middle| \frac{p^2}{\lambda} + \sqrt{\mu}V_{\sqrt{\mu}} \middle| \alpha \right\rangle + \int \left(2\varepsilon p^2 - 1\right)|\hat{\alpha}(p)|^2\,\mathrm{d}p \\
&\geq \varepsilon \int |\hat{\alpha}(p)|^2(1+p^2)\,\mathrm{d}p + \int (\varepsilon p^2 - \varepsilon - 1)|\hat{\alpha}(p)|^2\,\mathrm{d}p \\
&\geq \varepsilon \|\alpha\|_{H^1}^2 - A,
\end{aligned}$$

where we introduced $\varepsilon = \frac{1}{2} - \frac{1}{2\lambda} > 0$ and $A = \frac{1}{4}\int \left[\varepsilon p^2 - 1 - \varepsilon\right]_- \mathrm{d}p < \infty$. Since $\mathcal{F}^{1,\sqrt{\mu}V_{\sqrt{\mu}}}(0) = 0$ we get for the minimiser that $\left\|\alpha_{1,\sqrt{\mu}V_{\sqrt{\mu}}}\right\|_{H^1}$ is bounded uniformly in μ. We conclude that

$$\left\|\alpha_{\mu,V}\right\|_{H^1} \leq C\mu^{3/4}. \qquad \square$$

We now use this to bound $\Delta_{\mu,V} = 2\widehat{V\alpha_{\mu,V}} = 2(2\pi)^{-3/2}\hat{V} * \hat{\alpha}_{\mu,V}$. Note that since $\hat{V} \leq 0$, we have that $\Delta_{\mu,V} \leq 0$. (In lemma 5.3 above we even showed that $\Delta_{\mu,V} < 0$.) First, we need the following.

Proposition 5.12. *Let $g \in H^1(\mathbb{R}^n)$, then $\|\hat{g}\|_{L^r} \leq C\|g\|_{H^1}$ for all $\frac{2n}{n+2} < r \leq 2$.*

In our case, of dimension $n = 3$, we thus get the condition $\frac{6}{5} < r \leq 2$.

Proof. For $r = 2$ this is clear. For $r < 2$ we compute

$$\|\hat{g}\|_{L^r}^r = \int |\hat{g}(p)|^r \frac{(1+p^2)^{r/2}}{(1+p^2)^{r/2}}\,\mathrm{d}p \leq \left(\int |\hat{g}(p)|^2\left(1+p^2\right) \mathrm{d}p\right)^{r/2}\left(\int \frac{1}{(1+p^2)^{\frac{r}{2-r}}}\,\mathrm{d}p\right)^{1-r/2}.$$

Since the second integral is finite for $\frac{2r}{2-r} > n$ we conclude the desired. $\square$

With this we may bound $\Delta_{\mu,V}$.

Proposition 5.13. *The function* $\Delta_{\mu,V}$ *satisfies*

$$|\Delta_{\mu,V}(p)| \leq C\mu^{3/4} \qquad \text{and} \qquad |\Delta_{\mu,V}(p') - \Delta_{\mu,V}(p)| \leq C\mu^{3/4}|p'-p|.$$

Proof. We compute

$$|\Delta_{\mu,V}(p)| \leq \frac{2}{(2\pi)^{3/2}} \int \left|\hat{V}(p-q)\right| \hat{\alpha}_{\mu,V}(q)\,\mathrm{d}q \leq C \left\|\hat{V}\right\|_{L^3} \left\|\hat{\alpha}_{\mu,V}\right\|_{L^{3/2}} \leq C \left\|V\right\|_{L^{3/2}} \left\|\alpha_{\mu,V}\right\|_{H^1} \leq C\mu^{3/4}$$

by the Hausdorff-Young inequality [19, Thm. 5.7] and the bounds above. The bound for the difference is similar, using that

$$\begin{aligned}\left\|\hat{V}(p'-\cdot) - \hat{V}(p-\cdot)\right\|_{L^3} &\leq C\left(\int \left|e^{-ip'x} - e^{-ipx}\right|^{3/2} |V(x)|^{3/2}\,\mathrm{d}x\right)^{2/3} \\ &\leq C\left(\int |p'-p|^{3/2}\,|x|^{3/2}|V(x)|^{3/2}\,\mathrm{d}x\right)^{2/3} \\ &= C\,\|V|\cdot|\|_{L^{3/2}}\,|p'-p|,\end{aligned}$$

where we used that $\hat{V}(p'-\cdot) - \hat{V}(p-\cdot)$ is the Fourier transform of $\left(e^{-ip'x} - e^{-ipx}\right) V(-x)$. □

For the sake of simplifying notation, we will just write Δ for the function $\Delta_{\mu,V}$ from now on.

Note that this is not a very good bound. One would think that Δ vanishes exponentially as $\mu \to 0$. (We prove this for the specific value $\Delta(\sqrt{\mu})$ below.) This bound is however good enough to get some reasonable control on $\tilde{m}_\mu(\Delta)$. This is what we now do.

By computing the spherical part of the integral, splitting the integral according to $p^2 < 2\mu$ and $p^2 > 2\mu$, and using the substitutions $s = \frac{\mu - p^2}{\mu}$ and $s = \frac{p^2-\mu}{\mu}$ we may rewrite $\tilde{m}_\mu(\Delta)$ as

$$\begin{aligned}\tilde{m}_\mu(\Delta) = \frac{\sqrt{\mu}}{4\pi^2}\Bigg[&\int_0^1 \frac{\sqrt{1-s}-1}{\sqrt{s^2 + \left(\frac{\Delta(\sqrt{\mu}\sqrt{1-s})}{\mu}\right)^2}} + \frac{\sqrt{1+s}-1}{\sqrt{s^2 + \left(\frac{\Delta(\sqrt{\mu}\sqrt{1+s})}{\mu}\right)^2}} - \frac{1}{\sqrt{1-s}} - \frac{1}{\sqrt{1+s}}\,\mathrm{d}s \\ &+ \int_0^1 \frac{1}{\sqrt{s^2 + \left(\frac{\Delta(\sqrt{\mu}\sqrt{1-s})}{\mu}\right)^2}} + \frac{1}{\sqrt{s^2 + \left(\frac{\Delta(\sqrt{\mu}\sqrt{1+s})}{\mu}\right)^2}}\,\mathrm{d}s \\ &+ \int_1^\infty \frac{\sqrt{1+s}}{\sqrt{s^2 + \left(\frac{\Delta(\sqrt{\mu}\sqrt{1+s})}{\mu}\right)^2}} - \frac{1}{\sqrt{1+s}}\,\mathrm{d}s\Bigg].\end{aligned}$$

We claim that

$$\begin{aligned}\tilde{m}_\mu(\Delta) = \frac{\sqrt{\mu}}{4\pi^2}\Bigg[&\int_0^1 \frac{\sqrt{1-s}-1}{\sqrt{s^2 + \left(\frac{\Delta(\sqrt{\mu})}{\mu}\right)^2}} + \frac{\sqrt{1+s}-1}{\sqrt{s^2 + \left(\frac{\Delta(\sqrt{\mu})}{\mu}\right)^2}} - \frac{1}{\sqrt{1-s}} - \frac{1}{\sqrt{1+s}}\,\mathrm{d}s \\ &+ \int_0^1 \frac{2}{\sqrt{s^2 + \left(\frac{\Delta(\sqrt{\mu})}{\mu}\right)^2}}\,\mathrm{d}s + \int_1^\infty \frac{\sqrt{1+s}}{\sqrt{s^2 + \left(\frac{\Delta(\sqrt{\mu})}{\mu}\right)^2}} - \frac{1}{\sqrt{1+s}}\,\mathrm{d}s + o(1)\Bigg]. \qquad (5.8)\end{aligned}$$

The first and last integrals follow by dominated convergence. For the second integral we give a sketch of why it should be true. Define the function(s) $x(s) = \frac{\Delta(\sqrt{\mu}\sqrt{1\pm s})}{\mu}$. We then want to show that

$$\int_0^1 \frac{1}{\sqrt{s^2 + x(s)^2}} - \frac{1}{\sqrt{s^2 + x(0)^2}}\,\mathrm{d}s \to 0.$$

As in the proof of lemma 5.8 we first want to show that $|x(s)| \le C|x(0)|$. This is enough, as we prove below. Our attempts at proving such bound have however been unsuccessful. We give an argument for why, one should have such a bound.

The minimiser $\hat{\alpha}_{\mu,V}$ is more or less localised at Ω_μ, (where $\hat{\alpha}_{\mu,V}(\sqrt{\mu}) = \frac{1}{2}$) meaning that $\hat{\alpha}_{\mu,V}(p)$ decays rapidly in $\mathrm{dist}(p, \Omega_\mu)$. (To get an intuition for this, it is perhaps most easily seen for the minimiser $\alpha_{1,\sqrt{\mu}V_{\sqrt{\mu}}}$.) We have by definition for $p = O(\sqrt{\mu})$ that

$$\Delta(p) = \frac{2}{(2\pi)^{3/2}} \int \hat{V}(p-q)\hat{\alpha}_{\mu,V}(q)\,\mathrm{d}q \simeq \frac{2}{(2\pi)^{3/2}} \int_{|q|<\varepsilon} \hat{V}(0)\hat{\alpha}_{\mu,V}(q)\,\mathrm{d}q,$$

for some absolute constant $\varepsilon > 0$ by the continuity of $\hat{V}$ and the rapid decay of $\hat{\alpha}_{\mu,V}$. We have not been able to explicitly bound the error term in this approximation. Intuitively, we have that $\hat{\alpha}_{\mu,V}(q)$ is very small as soon as $|q| \gg \sqrt{\mu}$. Here the error-term has $|q| \ge \varepsilon = \Theta(1) \gg \sqrt{\mu}$, so indeed, the error should be small. This approximation does not depend on p, and so $\Delta(p) \sim \Delta(\sqrt{\mu})$ are of the same order. Hence $|x(s)| \le C|x(0)|$ and so

$$\left| \frac{1}{\sqrt{s^2 + x(s)^2}} - \frac{1}{\sqrt{s^2 + x(0)^2}} \right| \le C\mu^{1/4} \frac{|x(0)|}{\sqrt{s^2 + x(0)^2}\left(s + \sqrt{s^2 + x(0)^2}\right)},$$

where we used that $|x(s) - x(0)| \le C\mu^{1/4}s$ by the Lipschitz bound on Δ and a computation similarly as in the proof of lemma 5.8. Now, one may compute that

$$\int_0^1 C\mu^{1/4} \frac{|x(0)|}{\sqrt{s^2 + x(0)^2}\left(s + \sqrt{s^2 + x(0)^2}\right)}\,\mathrm{d}s = O(\mu^{1/4}).$$

Thus, this also vanishes. We conclude that equation (5.8) holds.

The remainder of this section is quite similar to section 4.3. We decompose

$$B_\Delta := V^{1/2}\frac{1}{E_\Delta}|V|^{1/2} = V^{1/2}\frac{1}{p^2}|V|^{1/2} + \tilde{m}_\mu(\Delta)\left|V^{1/2}\right\rangle\left\langle|V|^{1/2}\right| + A_{\Delta,\mu},$$

where $A_{\Delta,\mu}$ is defined such that this holds. That is, its kernel is

$$A_{\Delta,\mu}(x,y) = V(x)^{1/2}|V(y)|^{1/2}\frac{1}{2\pi^2}\int_0^\infty \left(\frac{\sin p|x-y|}{p|x-y|} - 1\right)\left(\frac{1}{E_\Delta(p)} - \frac{1}{p^2}\right)p^2\,\mathrm{d}p.$$

In section 5.1 we saw that B_Δ has -1 as its lowest eigenvalue. Now,

Proposition 5.14. *In the limit $\mu \to 0$ we have $\Delta(\sqrt{\mu}) = o(\mu)$.*

Proof. Suppose for contradiction that $\theta := \frac{-\Delta(\sqrt{\mu})}{\mu}$ does not vanish. That is, suppose that there is some subsequence with $\theta > B$ for $\mu \to 0$ for some constant $B > 0$. We use the decomposition

$$B_\Delta = V^{1/2}\frac{1}{p^2}|V|^{1/2} + \tilde{m}_\mu(\Delta)\left|V^{1/2}\right\rangle\left\langle|V|^{1/2}\right| + A_{\Delta,\mu}.$$

By the assumptions on V, we have that the spectrum of $V^{1/2}\frac{1}{p^2}|V|^{1/2}$ is contained in $(-1,\infty)$. We show that the remaining two terms in the decomposition above vanish in the limit $\mu \to 0$, and so that the spectrum of B_Δ approaches that of $V^{1/2}\frac{1}{p^2}|V|^{1/2}$. Since the latter has its lowest eigenvalue strictly larger than -1, we get a contradiction.

For $\tilde{m}_\mu(\Delta)$ we use equation (5.8) above. The only term that does not immediately vanish in the limit $\mu \to 0$ is the term

$$\frac{\mu^{1/2}}{4\pi^2}\int \frac{\sqrt{1+s}}{\sqrt{s^2+\theta^2}} - \frac{1}{\sqrt{1+s}}\,\mathrm{d}s.$$

By splitting this integral according to $s < \theta$ and $s > \theta$ we see that this term may be bounded by $C\mu^{1/2}\theta \le C\mu^{1/4}$ by proposition 5.13. Hence this term indeed also vanishes.

For the kernel of $A_{\Delta,\mu}$ we use that $\left|\frac{\sin b}{b} - 1\right| \le Cb^\gamma$ for any $0 \le \gamma \le 2$ for the specific choice of $\gamma = \frac{1}{2}$. Then

$$\left|A_{\Delta,\mu}(x,y)\right| \le C|V(x)|^{1/2}|V(y)|^{1/2}|x-y|^{1/2}\left[\int_0^{\sqrt{2\mu}}\left|\frac{1}{E_\Delta}-\frac{1}{p^2}\right|p^{5/2}\,\mathrm{d}p + \int_{\sqrt{2\mu}}^\infty\left|\frac{1}{E_\Delta}-\frac{1}{p^2}\right|p^{5/2}\,\mathrm{d}p\right].$$

For the first integral we bound $E_\Delta(p) \ge |\Delta(p)| \ge B\mu$ and so

$$\int_0^{\sqrt{2\mu}}\left|\frac{1}{E_\Delta}-\frac{1}{p^2}\right|p^{5/2}\,\mathrm{d}p \le \int_0^{\sqrt{2\mu}}\frac{1}{B\mu}(2\mu)^{5/4} + (2\mu)^{1/4}\,\mathrm{d}p \le C\mu^{3/4}.$$

We bound the second integral as follows. First, with the substitution $s = \frac{p^2-\mu}{\mu}$ we have

$$\begin{aligned}
\int_{\sqrt{2\mu}}^\infty\left|\frac{1}{E_\Delta}-\frac{1}{p^2}\right|p^{5/2}\,\mathrm{d}p &= \frac{\mu^{3/4}}{2}\int_1^\infty\left|\frac{1+s}{\sqrt{s^2+\left(\frac{\Delta(\sqrt{\mu}\sqrt{1+s})}{\mu}\right)^2}}-1\right|\frac{1}{(1+s)^{1/4}}\,\mathrm{d}s \\
&\le \frac{\mu^{3/4}}{2}\int_1^\infty \frac{1}{s(1+s)^{1/4}} + \frac{\sqrt{s^2+\left(\frac{\Delta(\sqrt{\mu}\sqrt{1+s})}{\mu}\right)^2}-s}{s(1+s)^{1/4}}\,\mathrm{d}s \\
&\le C\mu^{3/4} + C\mu^{3/4}\int_1^\infty \frac{\left|\frac{\Delta(\sqrt{\mu}\sqrt{1+s})}{\mu}\right|}{s(1+s)^{1/4}}\,\mathrm{d}s \\
&\le C\mu^{1/2},
\end{aligned}$$

where we used that $|\Delta(p)| \le C\mu^{3/4}$. The integral $\iint |V(x)||V(y)||x-y|\,\mathrm{d}x\,\mathrm{d}y < \infty$ is finite by the assumptions on V. Thus $\left\|A_{\Delta,\mu}\right\|_2 \le C\mu^{1/2}$ vanishes as desired. □

Using this refined bound, $\theta = \frac{-\Delta(\sqrt{\mu})}{\mu} = o(1)$, we may use a dominated convergence argument to show that

$$\begin{aligned}
\tilde{m}_\mu(\Delta) = \frac{\sqrt{\mu}}{4\pi^2}\Bigg[&\int_0^1 \frac{\sqrt{1-s}-1}{s} + \frac{\sqrt{1+s}-1}{s} - \frac{1}{\sqrt{1-s}} - \frac{1}{\sqrt{1+s}}\,\mathrm{d}s \\
&+ \int_0^1 \frac{2}{\sqrt{s^2+\left(\frac{\Delta(\sqrt{\mu})}{\mu}\right)^2}}\,\mathrm{d}s + \int_1^\infty \frac{\sqrt{1+s}}{s} - \frac{1}{\sqrt{1+s}}\,\mathrm{d}s + o(1)\Bigg].
\end{aligned}$$

This expression is the same as in section 5.1. We conclude that

$$\tilde{m}_\mu(\Delta) = \frac{\sqrt{\mu}}{2\pi^2}\left(\log\frac{\mu}{-\Delta(\sqrt{\mu})} - 2 + \log 8 + o(1)\right)$$

in the limit $\mu \to 0$. In particular $\tilde{m}_\mu(\Delta) \gg \sqrt{\mu}$. Now, we are interested in bounding $A_{\Delta,\mu}$.

Proposition 5.15. *We have*

$$\lim_{\mu\to 0}\frac{\left\|A_{\Delta,\mu}\right\|_2}{\tilde{m}_\mu(\Delta)} = 0.$$

Proof. The proof is similar as above, only we give a more refined bound on the kernel. We bound the $\frac{\sin b}{b}$ term by

$$\left|\frac{\sin|p||x-y|}{|p||x-y|} - 1\right|$$
$$\leq C\left[p^2 Z^2 1_{\{|x-y|<Z\}} + |p|^{1/2}|x-y|^{1/2}1_{\{|x-y|>Z\}}\right]1_{\{p^2<2\mu\}} + C|p|^{1/2}|x-y|^{1/2}1_{\{p^2>2\mu\}}.$$

Where $Z > 0$ is arbitrary, and the constant C does not depend on Z. Then

$$\left|A_{\Delta,\mu}(x,y)\right| \leq C|V(x)|^{1/2}|V(y)|^{1/2}\left[Z^2\int_0^{\sqrt{2\mu}}\left|\frac{1}{E_\Delta} - \frac{1}{p^2}\right|p^4\,\mathrm{d}p\right.$$
$$\left.+|x-y|^{1/2}1_{\{|x-y|>Z\}}\int_0^{\sqrt{2\mu}}\left|\frac{1}{E_\Delta} - \frac{1}{p^2}\right|p^{5/2}\,\mathrm{d}p + |x-y|^{1/2}\int_{\sqrt{2\mu}}^\infty\left|\frac{1}{E_\Delta} - \frac{1}{p^2}\right|p^{5/2}\,\mathrm{d}p\right].$$

Now, the first and second integral may be bounded by $\tilde{m}_\mu(\Delta)\mu$ and $\tilde{m}_\mu(\Delta)\mu^{1/4}$ exactly as in section 4.3. For any α we have

$$\int_0^{\sqrt{2\mu}}\left|\frac{1}{E_\Delta} - \frac{1}{p^2}\right|p^\alpha\,\mathrm{d}p \leq \int_0^{\sqrt{2\mu}}\left(\frac{1}{E_\Delta} - \frac{1}{p^2}\right)p^\alpha + 2p^{\alpha-2}\,\mathrm{d}p \leq C\tilde{m}_\mu(\Delta)\mu^{\alpha/2} + C\mu^{\frac{\alpha-1}{2}} \leq C\tilde{m}_\mu(\Delta)\mu^{\frac{\alpha-2}{2}}.$$

Similarly as before, the last integral may be bounded by $\mu^{1/2} \ll \tilde{m}_\mu(\Delta)$. Thus we get

$$\lim_{\mu\to 0}\frac{\left\|A_{\Delta,\mu}\right\|_2}{\tilde{m}_\mu(\Delta)} = 0. \qquad \square$$

We may decompose

$$1 + B_\Delta = \left(1 + V^{1/2}\frac{1}{p^2}|V|^{1/2}\right)\left(1 + \frac{\tilde{m}_\mu(\Delta)}{1 + V^{1/2}\frac{1}{p^2}|V|^{1/2}}\left(\left|V^{1/2}\right\rangle\left\langle|V|^{1/2}\right| + \frac{A_{\Delta,\mu}}{\tilde{m}_\mu(\Delta)}\right)\right).$$

Since -1 is an eigenvalue of B_Δ we get that -1 is an eigenvalue of

$$\frac{\tilde{m}_\mu(\Delta)}{1 + V^{1/2}\frac{1}{p^2}|V|^{1/2}}\left(\left|V^{1/2}\right\rangle\left\langle|V|^{1/2}\right| + \frac{A_{\Delta,\mu}}{\tilde{m}_\mu(\Delta)}\right).$$

Proposition 5.15 above gives that the term $\frac{A_{\Delta,\mu}}{\tilde{m}_\mu(\Delta)}$ vanishes in the limit $\mu \to 0$. The other term has rank one and thus we get

$$\lim_{\mu\to 0}\frac{-1}{\tilde{m}_\mu(\Delta)} = \left\langle|V|^{1/2}\left|\frac{1}{1 + V^{1/2}\frac{1}{p^2}|V|^{1/2}}\right|V^{1/2}\right\rangle = 4\pi a.$$

We now show, that this convergence is $o(\mu^{1/2})$.

First, we improve on proposition 5.15. Since $\tilde{m}_\mu(\Delta)$ is of order 1 in the limit $\mu \to 0$ we have for the third integral in the proof of proposition 5.15 that

$$\int_{\sqrt{2\mu}}^{\infty} \left| \frac{1}{E_\Delta} - \frac{1}{p^2} \right| p^{5/2} \, \mathrm{d}p \leq C\mu^{1/2} \ll \mu^{1/4} \tilde{m}_\mu(\Delta).$$

Hence we have for any $Z > 0$ and a constant C, that does not depend on Z that

$$\limsup_{\mu \to 0} \frac{\|A_{\Delta,\mu}\|_2}{\mu^{1/4}\tilde{m}_\mu(\Delta)} \leq C \left(\iint_{\{|x-y|>Z\}} |V(x)||V(y)||x-y| \, \mathrm{d}x \, \mathrm{d}y \right)^{1/2}.$$

Taking $Z \to \infty$ we get that

$$\lim_{\mu \to 0} \frac{\|A_{\Delta,\mu}\|_2}{\mu^{1/4}\tilde{m}_\mu(\Delta)} = 0.$$

In particular, $A_{\Delta,\mu}$ vanishes in the limit $\mu \to 0$. Thus the operator

$$1 + V^{1/2} \frac{1}{p^2} |V|^{1/2} + A_{\Delta,\mu}$$

is invertible for small μ and so we may write

$$1 + B_\Delta = \left(1 + V^{1/2} \frac{1}{p^2} |V|^{1/2} + A_{\Delta,\mu} \right) \left(1 + \frac{\tilde{m}_\mu(\Delta)}{1 + V^{1/2} \frac{1}{p^2} |V|^{1/2} + A_{\Delta,\mu}} \left| V^{1/2} \right\rangle \left\langle |V|^{1/2} \right| \right).$$

Since -1 is an eigenvalue of B_Δ we get that -1 is an eigenvalue of the latter operator. This has rank one and so we get that

$$\frac{-1}{\tilde{m}_\mu(\Delta)} = \left\langle |V|^{1/2} \middle| \frac{1}{1 + V^{1/2} \frac{1}{p^2} |V|^{1/2} + A_{\Delta,\mu}} \middle| V^{1/2} \right\rangle. \tag{5.9}$$

We decompose the middle operator on the right-hand-side as

$$\begin{aligned} \frac{1}{1 + V^{1/2} \frac{1}{p^2} |V|^{1/2} + A_{\Delta,\mu}} \quad = \quad & \frac{1}{1 + V^{1/2} \frac{1}{p^2} |V|^{1/2}} - \frac{1}{1 + V^{1/2} \frac{1}{p^2} |V|^{1/2}} A_{\Delta,\mu} \frac{1}{1 + V^{1/2} \frac{1}{p^2} |V|^{1/2}} \\ & + \frac{1}{1 + V^{1/2} \frac{1}{p^2} |V|^{1/2}} A_{\Delta,\mu} \frac{1}{1 + V^{1/2} \frac{1}{p^2} |V|^{1/2} + A_{\Delta,\mu}} A_{\Delta,\mu} \frac{1}{1 + V^{1/2} \frac{1}{p^2} |V|^{1/2}}. \end{aligned}$$

Plugging this into equation (5.9) above we get $4\pi a$ for the first term. The second term gives

$$\left\langle f \middle| \operatorname{sgn} V A_{\Delta,\mu} \middle| f \right\rangle, \qquad \text{with} \qquad f = \frac{1}{1 + V^{1/2} \frac{1}{p^2} |V|^{1/2}} V^{1/2}.$$

The third term is $o(\mu^{1/2})$ by the bound on $A_{\Delta,\mu}$ above. We show that the second term is $o(\mu^{1/2})$ as well.

Proposition 5.16. *In the limit $\mu \to 0$ we have $\left\langle f \middle| \operatorname{sgn} V A_{\Delta,\mu} \middle| f \right\rangle = o(\mu^{1/2})$.*

Proof. This is similar to the bound on $A_{\Delta,\mu}$ above. We bound the kernel of $A_{\Delta,\mu}$ by

$$|A_{\Delta,\mu}(x,y)| \leq C|V(x)|^{1/2}|V(y)|^{1/2}\left[Z^2\int_0^{\sqrt{2\mu}}\left|\frac{1}{E_\Delta}-\frac{1}{p^2}\right|p^4\,\mathrm{d}p\right.$$
$$\left.+|x-y|1_{\{|x-y|>Z\}}\int_0^{\sqrt{2\mu}}\left|\frac{1}{E_\Delta}-\frac{1}{p^2}\right|p^3\,\mathrm{d}p+|x-y|^{3/4}\int_{\sqrt{2\mu}}^\infty\left|\frac{1}{E_\Delta}-\frac{1}{p^2}\right|p^{11/4}\,\mathrm{d}p\right].$$

These integrals are bounded by $\mu, \mu^{1/2}$ and $\mu^{5/8}$ respectively similarly as in proposition 5.15. (recall that $\tilde{m}_\mu(\Delta)$ is of order 1.) Thus

$$\limsup_{\mu\to 0}\frac{\left|\left\langle f\middle|\operatorname{sgn} V A_{\Delta,\mu}\middle|f\right\rangle\right|}{\mu^{1/2}} \leq C\iint_{\{|x-y|>Z\}}|f(x)||V(x)|^{1/2}|x-y||f(y)||V(y)|^{1/2}\,\mathrm{d}x\,\mathrm{d}y.$$

In proposition 4.14 above it was proved that the integrand here is indeed integrable. Hence by taking $Z\to\infty$ we get the desired. □

We thus conclude that

$$\tilde{m}_\mu(\Delta)=\frac{-1}{4\pi a}+o\left(\mu^{1/2}\right).$$

With the asymptotics of $\tilde{m}_\mu(\Delta)$ above we thus get

$$\lim_{\mu\to 0}\left(\log\frac{\mu}{-\Delta(\sqrt{\mu})}+\frac{\pi}{2\sqrt{\mu}a}\right)=2-\log 8.$$

Now, we want to replace $-\Delta(\sqrt{\mu})$ by the energy gap $\Xi=\inf E_\Delta$. Clearly $\Xi\leq-\Delta(\sqrt{\mu})$. On the other hand

$$\Xi\geq\min_{|p^2-\mu|\leq\Xi}-\Delta(p)\geq-\Delta(\sqrt{\mu})(1+o(1))$$

Thus we conclude the desired

$$\lim_{\mu\to 0}\left(\log\frac{\mu}{\Xi}+\frac{\pi}{2\sqrt{\mu}a}\right)=2-\log 8.$$

This concludes the proof of theorem 5.9.

6 Omitting the Direct and Exchange Terms

In this section we consider the validity of omitting the direct and exchange terms. We will consider a sequence of short-range potentials, converging to a point interaction and show that in some sense, the direct and exchange terms can be seen as a renormalisation of the chemical potential in this limit. Morally this means that the approximation of omitting the terms is good for short-range potentials. This section is based on [6]. First we state the results for the model including the direct and exchange terms.

6.1 Statement of Results

Similarly as in section 2, by consider translation-invariant states and doing a formal infinite volume expansion, one is led to the functional [6]

$$\mathcal{F}_T^V(\Gamma) = \int (p^2-\mu)\hat{\gamma}(p)\,\mathrm{d}p + \int |\alpha(x)|^2 V(x)\,\mathrm{d}x - TS(\Gamma) - \int |\gamma(x)|^2 V(x)\,\mathrm{d}x + 2\gamma(0)^2 \int V(x)\,\mathrm{d}x, \tag{6.1}$$

for Γ of the form

$$\Gamma(p) = \begin{pmatrix} \hat{\gamma}(p) & \hat{\alpha}(p) \\ \hat{\alpha}(p) & 1-\hat{\gamma}(-p) \end{pmatrix}, \qquad 0 \le \Gamma \le 1.$$

We will need the concept of a normal state, Γ_0, which is a minimiser of $\mathcal{F}_T^V$ when restricted to states with $\alpha \equiv 0$. They are of the form

$$\hat{\gamma}_0(p) = \frac{1}{1+\exp\left(\frac{\varepsilon^{\gamma_0}(p)-\tilde{\mu}^{\gamma_0}}{T}\right)},$$

where for general γ we have introduced

$$\varepsilon^{\gamma}(p) = p^2 - \frac{2}{(2\pi)^{3/2}} \int \left(\hat{V}(p-q)-\hat{V}(0)\right)\hat{\gamma}(q)\,\mathrm{d}q$$
$$\tilde{\mu}^{\gamma} = \mu - \frac{2}{(2\pi)^{3/2}}\hat{V}(0)\int \hat{\gamma}(p)\,\mathrm{d}p$$

some sort of renormalised kinetic energy and chemical potential.

As before, we will say that a system is in a superconducting state if such a normal state Γ_0 is not a minimiser of $\mathcal{F}_T^V$. We have the following analogue of theorem 3.1.

Proposition 6.1 ([6, Prop. 1])**.** *Let $\mu \in \mathbb{R}$, $0 \le T < \infty$ and let $V \in L^1 \cap L^{3/2}$ be real-valued and reflection-symmetric with $\left\|\hat{V}\right\|_{L^\infty} \le 2\hat{V}(0)$. Then $\mathcal{F}_T^V$ is bounded from below and attains a minimiser $\Gamma = (\gamma, \alpha)$ on*

$$\mathcal{D} := \left\{ \Gamma(p) = \begin{pmatrix} \hat{\gamma}(p) & \hat{\alpha}(p) \\ \hat{\alpha}(p) & 1-\hat{\gamma}(-p) \end{pmatrix} : \hat{\gamma} \in L^1(\mathbb{R}^3,(1+p^2)\,\mathrm{d}p), \alpha \in H^1(\mathbb{R}^3,\mathrm{d}x), 0 \le \Gamma \le 1 \right\}.$$

Moreover, the minimising α satisfies the BCS gap equation

$$(K_{T,\mu}^{\gamma,\Delta} + V)\alpha = 0,$$

where

$$K_{T,\mu}^{\gamma,\Delta} := \frac{E_\mu^{\gamma,\Delta}}{\tanh\left(\frac{E_\mu^{\gamma,\Delta}}{2T}\right)}, \qquad E_\mu^{\gamma,\Delta} := \sqrt{(\varepsilon^\gamma - \tilde{\mu}^\gamma)^2 + |\Delta|^2}, \qquad \Delta := 2\widehat{V\alpha} = 2(2\pi)^{-3/2}\hat{V} * \hat{\alpha}.$$

Remark 6.2. Additionally, for the minimiser $\Gamma = (\gamma, \alpha)$, γ satisfies the Euler-Lagrange equation

$$\hat{\gamma}(p) = \frac{1}{2} - \frac{\varepsilon^\gamma(p) - \tilde{\mu}^\gamma}{2K_{T,\mu}^{\gamma,\Delta}(p)}.$$

Also, we have that Δ satisfies a BCS gap equation for Δ as well

$$-\Delta(p) = \frac{1}{(2\pi)^{3/2}} \int \hat{V}(p-q)\frac{\Delta(q)}{K_{T,\mu}^{\gamma,\Delta}(q)}\,\mathrm{d}q.$$

Remark 6.3. Note here the assumption $\hat{V}(0) \geq \left\|\hat{V}\right\|_{L^\infty} \geq 0$, which is in sharp contrast to the assumptions in section 5. This assumption is needed to ensure stability of the system.

Similarly we have the following analogue of theorem 3.7.

Theorem 6.4 ([6, Thm. 1])**.** *Let $\mu \in \mathbb{R}$, $0 \leq T < \infty$ and let $V \in L^1 \cap L^{3/2}$ be real-valued and reflection-symmetric with $\left\|\hat{V}\right\|_{L^\infty} \leq 2\hat{V}(0)$. Let Γ_0 be a normal state.*

- *If* $\inf \operatorname{spec}(K_{T,\mu}^{\gamma_0,0} + V) < 0$, *then Γ_0 is unstable, i.e.* $\inf_{\Gamma \in \mathcal{D}} \mathcal{F}_T^V(\Gamma) < \mathcal{F}_T^V(\Gamma_0)$.
- *If Γ_0 is unstable, then there exists $\Gamma = (\gamma, \alpha) \in \mathcal{D}$ with $\alpha \not\equiv 0$ satisfying the BCS gap equation* $(K_{T,\mu}^{\gamma,\Delta} + V)\alpha = 0$.

Remark 6.5. Note that this is "weaker" than theorem 3.7. We lack the implication

$$\alpha \not\equiv 0 \text{ satisfies the BCS gap equation} \implies \Gamma_0 \text{ is unstable.}$$

The proof of that implication in theorem 3.7 does not work here, since $K_{T,\mu}^{\gamma,\Delta}$ depend on γ in a complicated way. The same argument as in the proof of theorem 3.7 gives that $K_{T,\mu}^{\gamma,0} + V$ indeed does have a negative eigenvalue, but it is not clear how this should imply that $K_{T,\mu}^{\gamma_0,0} + V$ has a negative eigenvalue.

The proofs of these are similar to their counterparts (theorems 3.1 and 3.7) and can be found sketched in [6]. We will not provide them here. We will consider a sequence of potentials $\{V_\ell\}_{\ell>0}$ with $\ell \to 0$ satisfying the following assumptions

Assumption 6.6 ([6, Assumption 1])**.** We impose the following assumptions on the sequence $\{V_\ell\}$.

(A1) $V_\ell \in L^1 \cap L^2$ is real-valued and reflection-symmetric.

(A2) The range of V_ℓ is at most ℓ, meaning that $\operatorname{supp} V_\ell \subset B(0, \ell)$.

(A3) The scattering length $a(V_\ell) < 0$ is negative and satisfies $\lim_{\ell \to 0} a(V_\ell) = a < 0$.

(A4) The 1-norm of V_ℓ is bounded, i.e. $\limsup_{\ell \to 0} \|V_\ell\|_{L^1} < \infty$.

(A5) The average of V_ℓ is positive, meaning $\hat{V}_\ell(0) > 0$ and $\lim_{\ell \to 0} \hat{V}_\ell(0) = \mathcal{V} \geq 0$.

(A6) $\left\|\hat{V}_\ell\right\|_{L^\infty} \leq 2\hat{V}_\ell(0)$ for all $\ell > 0$.

(A7) There exists a constant $C_1 > 0$ and some $N \in \mathbb{N}$ with $\|V_\ell\|_{L^2} \leq C_1 \ell^{-N}$ for small ℓ.

(A8) There exists a $0 < b < 1$ such that $\inf \operatorname{spec}\left(p^2 + V_\ell - |p|^b\right) > C_2 > -\infty$ uniformly in ℓ.

(A9) The operator $1 + V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2}$ is invertible and has an eigenvalue e_ℓ of order ℓ with corresponding eigenvector ϕ_ℓ. (Meaning that there exists constants $0 < c < C$ with $c\ell < e_\ell < C\ell$.) Moreover, the operator

$$\frac{1}{1 + V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2}} (1 - P_\ell)$$

is uniformly bounded in ℓ, where $P_\ell = \frac{1}{\langle J_\ell \phi_\ell | \phi_\ell \rangle} |\phi_\ell\rangle \langle J_\ell \phi_\ell|$, and $J_\ell = \operatorname{sgn} V_\ell$.

(A10) The eigenvector ϕ_ℓ satisfies that

$$\frac{\left\langle |V_\ell|^{1/2} \middle| |\phi_\ell| \right\rangle}{|\langle \phi_\ell | \operatorname{sgn}(V_\ell) \phi_\ell \rangle|} = O(\ell^{1/2})$$

in the limit $\ell \to 0$.

Note that (A9) implies that the scattering length

$$a(V_\ell) := \frac{1}{4\pi} \left\langle |V_\ell|^{1/2} \middle| \frac{1}{1 + V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2}} \middle| V_\ell^{1/2} \right\rangle$$

is well-defined.

For examples of sequences of potentials satisfying these we refer to [6, Appendix A]. As described in [6] one should have in mind a sequence of potentials that are repulsive on lengthscales of order $\varepsilon_\ell \ll \ell$ and attractive up to length ℓ.

We now restrict to a sequence $\{V_\ell\}$ satisfying the assumptions above. We again define the renormalised kinetic energy and chemical potential

$$\varepsilon^{\gamma_\ell}(p) = p^2 - \frac{2}{(2\pi)^{3/2}} \int \left(\hat{V}_\ell(p-q) - \hat{V}_\ell(0) \right) \hat{\gamma}_\ell(q) \, \mathrm{d}q, \quad \tilde{\mu}^{\gamma_\ell} = \mu - \frac{2}{(2\pi)^{3/2}} \hat{V}_\ell(0) \int \hat{\gamma}_\ell(p) \, \mathrm{d}p.$$

Then

Theorem 6.7 (Effective Gap Equation)**.** *Let $T \geq 0$, let $\mu \in \mathbb{R}$ and let $\Gamma_\ell = (\gamma_\ell, \alpha_\ell)$ be a sequence of minimisers of $\mathcal{F}_T^{V_\ell}$ and define $\Delta_\ell := 2(2\pi)^{-3/2} \hat{V}_\ell * \hat{\alpha}_\ell$. Then there exists $\Delta \geq 0$ and $\hat{\gamma}$ with $|\Delta_\ell| \to \Delta$ pointwise, $\hat{\gamma}_\ell \to \hat{\gamma}$ pointwise and $\tilde{\mu}^{\gamma_\ell} \to \tilde{\mu}^\gamma$ as $\ell \to 0$, satisfying*

$$\hat{\gamma}(p) = \frac{1}{2} - \frac{p^2 - \tilde{\mu}}{2K_{T,\tilde{\mu}^\gamma}^{0,\Delta}}, \qquad \tilde{\mu}^\gamma = \mu - \frac{2\mathcal{V}}{(2\pi)^{3/2}} \int \hat{\gamma}(p) \, \mathrm{d}p \tag{6.2}$$

Moreover, if $\Delta_\ell \not\equiv 0$ for a subsequence of ℓ's then

$$\frac{-1}{4\pi a} = \frac{1}{(2\pi)^3} \int \frac{1}{K_{T,\tilde{\mu}^\gamma}^{0,\Delta}} - \frac{1}{p^2} \, \mathrm{d}p.$$

That is, Δ satisfies a sort of BCS Gap equation for numbers. The constant $\mathcal{V} = \lim_{\ell \to 0} \hat{V}_\ell(0)$ is as in (A5).

We define the critical temperature to be the T solving the BCS gap equation above for $\Delta = 0$. For this Δ we have that $\hat{\gamma}(p) = \frac{1}{1+\exp\left(\frac{p^2-\tilde{\mu}}{T}\right)}$. More precisely

Definition 6.8. Let $\mu > 0$. Then the *critical temperature* $T_c > 0$ and the *renormalised chemical potential* $\tilde{\mu} \in \mathbb{R}$ are given implicitly by

$$\frac{-1}{4\pi a} = \frac{1}{(2\pi)^3} \int \frac{\tanh\left(\frac{p^2-\tilde{\mu}}{2T_c}\right)}{p^2 - \tilde{\mu}} - \frac{1}{p^2} \, \mathrm{d}p, \qquad \tilde{\mu} = \mu - \frac{2\mathcal{V}}{(2\pi)^{3/2}} \int \frac{1}{1 + \exp\left(\frac{p^2-\tilde{\mu}}{T_c}\right)} \, \mathrm{d}p \tag{6.3}$$

Proposition 6.9. *The critical temperature and renormalised chemical potential exist and are uniquely given by equation* (6.3).

Proof. The defining equations can be written as

$$F(\tilde{\mu}, T_c) = \frac{-1}{4\pi a}, \qquad G(\tilde{\mu}, T_c) = \mu$$

where

$$F(\nu, T) = \frac{1}{(2\pi)^3} \int \frac{\tanh\left(\frac{p^2-\nu}{2T}\right)}{p^2 - \nu} - \frac{1}{p^2}\, \mathrm{d}p, \qquad G(\nu, T) = \nu + \frac{2\mathcal{V}}{(2\pi)^{3/2}} \int \frac{1}{1 + \exp\left(\frac{p^2-\nu}{T}\right)}\, \mathrm{d}p.$$

One may compute that [6]

$$\partial_T F < 0, \quad \partial_\nu F > 0, \qquad \partial_T G > 0, \quad \partial_\nu G > 0.$$

One also easily checks for any fixed $T > 0$ that

$$G(\nu, T) \to \infty \text{ as } \nu \to \infty \qquad \text{and} \qquad G(\nu, T) \to -\infty \text{ as } \nu \to -\infty.$$

Hence, the level set $\{G = \mu\}$ in (ν, T)-space is the graph of a strictly decreasing function, with limits $T \to \infty$ as $\nu \to -\infty$ and $T \to 0$ as $\nu \to \infty$. Thus it will intersect the graph of any strictly increasing function in exactly one point. The level set of F is indeed the graph of such a function. We conclude that T_c and $\tilde{\mu}$ are indeed well-defined and unique. □

Note that it is crucial that $\mu > 0$. If $\mu \leq 0$, then also $\tilde{\mu} \leq 0$ and hence the equation $F(\nu, T) = \frac{-1}{4\pi a}$ has no solution, since $F(\nu, T) < 0$ for any $\nu \leq 0$. That is, for $\mu \leq 0$ the critical temperature is $T_c = 0$.

Remark 6.10. In lemma 4.8 we studied the asymptotics of the left integral in equation (6.3). Hence we get the following asymptotic result. Compare to those of section 4.

Proposition 6.11. *As $a \to 0$ the critical temperature satisfies*

$$T_c = \tilde{\mu} \left(\frac{8}{\pi} e^{\gamma - 2} + o(1) \right) e^{\frac{\pi}{2\sqrt{\tilde{\mu}} a}}.$$

The proof is an easy corollary of lemma 4.8 once we note that $\mu > 0$ implies that $\tilde{\mu} > 0$ for small enough T_c.

The critical temperature defined above is a critical temperature in the following sense

Theorem 6.12. *Let $\mu \in \mathbb{R}$, let $T \geq 0$ and let $\Gamma_\ell^0 = (\gamma_\ell^0, 0)$ be a sequence of normal states for the functionals $\mathcal{F}_T^{V_\ell}$.*

(i) If $T < T_c$, then for sufficiently small ℓ the system is in a superconducting state, i.e. Γ_ℓ^0 is not a minimiser.

(ii) If $T > T_c$, then for sufficiently small ℓ the system is not in a superconducting, i.e. Γ_ℓ^0 is a minimiser.

6.2 Proofs

We give proofs of some of the technical lemmas needed to prove theorems 6.7 and 6.12 and prove these theorems. This section is based on [6] and all proofs, which we omit here, can be found there.

6.2.1 The Gap Equation

First we prove theorem 6.7.

Lemma 6.13 ([6, Lemma 3])**.** *There exists a constant* $C_1 > 0$ *such that*

$$\mathcal{F}_T^{V_\ell}(\Gamma) \geq -C_1 + \frac{1}{2}\int (1+p^2)(\hat{\gamma}-\hat{\gamma}_0)^2 \,\mathrm{d}p + \frac{1}{2}\int |p|^b |\hat{\alpha}(p)|^2 \,\mathrm{d}p$$

uniformly in ℓ*. Here* $\hat{\gamma}_0(p) = \frac{1}{1+\exp\left(\frac{p^2-\mu}{T}\right)}$.

Note that for a sequence of minimisers $(\gamma_\ell, \alpha_\ell)$ it follows that $\|\alpha_\ell\|_{L^2}$ is bounded uniformly in ℓ.

Lemma 6.14 ([6, Lemma 4])**.** *If* $\Gamma_\ell = (\gamma_\ell, \alpha_\ell)$ *is a sequence of minimisers of* $\mathcal{F}_T^{V_\ell}$ *then* $\int \hat{\gamma}_\ell(p)|p|^b \,\mathrm{d}p$ *is bounded uniformly in* ℓ*.*

It follows that $\|\hat{\gamma}_\ell\|_{L^1}$ is also bounded independently of ℓ. Since also $\|\hat{\gamma}_\ell\|_{L^\infty} \leq 1$ is bounded uniformly in ℓ we get that any L^p-norm of $\hat{\gamma}_\ell$ is uniformly bounded.

For the next lemma we first define

$$\tilde{m}_\mu^{\gamma_\ell,\Delta_\ell}(T) = \frac{1}{(2\pi)^3}\int \frac{1}{K_{T,\mu}^{\gamma_\ell,\Delta_\ell}} - \frac{1}{p^2}\,\mathrm{d}p.$$

Lemma 6.15 ([6, Lemma 5])**.** *Let* $\Gamma_\ell = (\gamma_\ell, \alpha_\ell)$ *be a sequence of minimisers of* $\mathcal{F}_T^{V_\ell}$*. Then there exists subsequences of* γ_ℓ *and* α_ℓ *(which we continue to denote by* γ_ℓ *and* α_ℓ*), a* $\gamma \in L^1 \cap L^\infty$ *and a* $\Delta \geq 0$ *such that in the limit* $\ell \to 0$

(i) $|\Delta_\ell|$ *converges pointwise to the constant function* Δ*,*

(ii) $\int \hat{\gamma}_\ell \,\mathrm{d}p$ *converges to* $\int \hat{\gamma}\,\mathrm{d}p$*,*

(iii) $\tilde{\mu}^{\gamma_\ell}$ *converges to* $\tilde{\mu}^\gamma := \mu - \frac{2\mathcal{V}}{(2\pi)^{3/2}}\int \hat{\gamma}\,\mathrm{d}p$*,*

(iv) $\varepsilon^{\gamma_\ell}$ *converges pointwise to the function* p^2*,*

(v) $\hat{\gamma}_\ell$ *converges pointwise to* $\hat{\gamma}$*,*

(vi) $\tilde{m}_\mu^{\gamma_\ell,\Delta_\ell}(T)$ *converges to* $\tilde{m}_\mu^{\gamma,\Delta}(T) = \tilde{m}_{\tilde{\mu}^\gamma}^{0,\Delta}(T)$*.*

Moreover, the limit $(\gamma, \tilde{\mu}^\gamma, \Delta)$ *satisfies equation* (6.2)*.*

In the proof of theorem 6.7 below, we will see that the result is actually true for the full sequence, and not only for a subsequence.

Proof. (i). First, $\check{\Delta}_\ell = 2V_\ell\alpha_\ell$. Thus $\left\|\check{\Delta}_\ell\right\|_{L^1} \leq 2\|V_\ell\|_{L^2}\|\alpha_\ell\|_{L^2} \leq C\ell^{-N}$ by the assumptions and lemma 6.13. We will now show that Δ_ℓ converges to a polynomial, and then, that this polynomial is of degree 0. Let

$$P_{\ell,N}(p) = \frac{1}{(2\pi)^{3/2}}\sum_{j=0}^{N}\frac{(-i)^j}{j!}\sum_{i_1,\ldots,i_j=1}^{3} c_{i_1,\ldots,i_j}^{(\ell,j)} p_{i_1}\cdots p_{i_j}$$

denote the N'th order Taylor polynomial of Δ_ℓ, expanded at $p = 0$. (Note that Δ_ℓ is infinitely often differentiable by the compact support of V_ℓ.) The coefficients $c_{i_1,\ldots,i_j}^{(\ell,j)}$ are given by

$$c_{i_1,\ldots,i_j}^{(\ell,j)} = (2\pi)^{3/2}\left(i\partial_{i_1}\right)\cdots\left(i\partial_{i_j}\right)\Delta_\ell(p)\Big|_{p=0} = \int_{\mathbb{R}^3}\check{\Delta}_\ell(x)x_{i_1}\cdots x_{i_j}\,\mathrm{d}x.$$

We may thus bound the error as

$$\left|\Delta_\ell(p) - P_{N,\ell}(p)\right| = \frac{1}{(2\pi)^{3/2}} \left| \int \check{\Delta}_\ell(x) \left(e^{-ipx} - \sum_{j=0}^{N} \frac{(-ipx)^j}{j!} \right) \mathrm{d}x \right| \leq \frac{1}{(2\pi)^{3/2}} \left\| \check{\Delta}_\ell \right\|_{L^1} |p|^{N+1} e^{\ell |p|} \ell^{N+1}$$

since $\check{\Delta}_\ell$ is supported in $B(0,\ell)$ and the sum in the integrand is the Taylor expansion of the exponential. By the bound above, this vanishes pointwise in p as $\ell \to 0$.

Define now $\overline{c_\ell} := \max_{0\leq j\leq N} \max_{1\leq i_1,\ldots,i_j\leq 3} \left\{ \left| c^{(\ell,j)}_{i_1,\ldots,i_j} \right| \right\}$ and $\overline{c} := \limsup_{\ell\to 0} \overline{c_\ell}$. If $\overline{c} = 0$ then $\Delta_\ell \to 0$ pointwise, and we are done. So suppose that $\overline{c} > 0$. Then, for some subsequence $\frac{P_{\ell,N}}{\overline{c_\ell}}$ converges pointwise to some polynomial P of degree $n \leq N$. In particular, also $\frac{\Delta_\ell}{\overline{c_\ell}}$ converges pointwise to P. We now show, that P must be of degree 0. Rewrite

$$2\hat{\alpha}_\ell = \frac{\Delta_\ell}{K^{\gamma_\ell,\Delta_\ell}_{T,\mu}} = \frac{\Delta_\ell}{E^{\gamma_\ell,\Delta_\ell}_\mu} \tanh\left(\frac{E^{\gamma_\ell,\Delta_\ell}_\mu}{2T} \right) = \frac{\Delta_\ell}{E^{\gamma_\ell,\Delta_\ell}_\mu} - \frac{\Delta_\ell}{E^{\gamma_\ell,\Delta_\ell}_\mu} \frac{2}{1 + \exp\left(\frac{E^{\gamma_\ell,\Delta_\ell}_\mu}{2T} \right)}.$$

Now, $E^{\gamma_\ell,\Delta_\ell}_\mu \geq \varepsilon^{\gamma_\ell} - \tilde{\mu}^{\gamma_\ell} \geq p^2 - \mu$ and $|\Delta_\ell| \leq E^{\gamma_\ell,\Delta_\ell}_\mu$. Thus, the second summand above has L^2-norm bounded uniformly in ℓ. Moreover,

$$|\varepsilon^{\gamma_\ell} - \tilde{\mu}^{\gamma_\ell}| \leq p^2 + |\mu| + \frac{2}{(2\pi)^{3/2}} \int \left(2\hat{V}_\ell(0) - \hat{V}_\ell(p-q) \right) \hat{\gamma}_\ell(q) \, \mathrm{d}q \leq p^2 + \nu,$$

where $\nu := |\mu| + \frac{8}{(2\pi)^{3/2}} \sup_{\ell>0} \hat{V}_\ell(0) \left\| \hat{\gamma}_\ell \right\|_{L^1}$ is finite by (A5) and lemma 6.14. In addition we thus have $E^{\gamma_\ell,\Delta_\ell}_\mu(p) \leq \sqrt{(p^2+\nu)^2 + |\Delta_\ell(p)|^2}$ and so

$$\left\| 2\hat{\alpha}_\ell \right\|_{L^2} \geq \left\| \frac{\Delta_\ell(p)}{\sqrt{(p^2+\nu)^2 + |\Delta_\ell(p)|^2}} \right\|_{L^2} - C.$$

We now consider the 2-norm on the right-hand-side. By dominated convergence we get for any $R > 0$ that

$$\limsup_{\ell\to 0} \int_{|p|\leq R} \frac{|\Delta_\ell(p)|^2}{(p^2+\nu)^2 + |\Delta_\ell(p)|^2} \, \mathrm{d}p = \int_{|p|\leq R} \frac{|\overline{c}P(p)|^2}{(p^2+\nu)^2 + |\overline{c}P(p)|^2} \, \mathrm{d}p$$

where the integrand in the latter integral should be replaced by 1 if $\overline{c} = \infty$. Now, if either $\overline{c} = \infty$ or $\deg P \geq 1$ this latter integral diverges as $R \to \infty$, contradicting that $\left\| \hat{\alpha}_\ell \right\|_{L^2}$ is bounded by lemma 6.13. We conclude that $\deg P = 0$ and $\overline{c} < \infty$. Then $\Delta_\ell \to \Delta := \overline{c}$ pointwise as desired.

(*ii*). Lemma 6.13 (or alternatively, the comment just after lemma 6.14) gives that $\hat{\gamma}_\ell$ is uniformly bounded in L^2. Thus, by Banach-Alaoglu we have that some subsequence $\hat{\gamma}_\ell$ converges to some $\hat{\gamma}$ weakly in L^2. Since $1_{B(0,R)} \in L^2$ for any $R > 0$ we thus have that

$$\lim_{\ell\to 0} \int_{B(0,R)} \hat{\gamma}_\ell \, \mathrm{d}p = \int_{B(0,R)} \hat{\gamma} \, \mathrm{d}p.$$

In particular for any $R > 0$ we have

$$\limsup_{\ell\to 0} \int_{\mathbb{R}^3} \hat{\gamma}_\ell \, \mathrm{d}p \geq \lim_{\ell\to 0} \int_{B(0,R)} \hat{\gamma}_\ell \, \mathrm{d}p = \int_{B(0,R)} \hat{\gamma} \, \mathrm{d}p.$$

Thus $\delta := \limsup_{\ell\to 0}\int_{\mathbb{R}^3}\hat\gamma_\ell\,\mathrm{d}p - \int_{\mathbb{R}^3}\hat\gamma\,\mathrm{d}p \geq 0$ is positive. We now prove that $\delta = 0$. By lemma 6.14 we have the following for any $R > 0$

$$\begin{aligned}\infty > C &\geq \limsup_{\ell\to 0}\int_{|p|\geq R}\hat\gamma_\ell(p)|p|^b\,\mathrm{d}p \geq R^b\limsup_{\ell\to 0}\int_{|p|\geq R}\hat\gamma_\ell(p)\,\mathrm{d}p\\ &= R^b\limsup_{\ell\to 0}\left(\int_{\mathbb{R}^3}\hat\gamma_\ell\,\mathrm{d}p - \int_{B(0,R)}\hat\gamma_\ell\,\mathrm{d}p\right) = R^b\left(\delta + \int_{\mathbb{R}^3}\hat\gamma\,\mathrm{d}p - \int_{B(0,R)}\hat\gamma\,\mathrm{d}p\right) \geq \delta R^b.\end{aligned}$$

Thus, taking $R\to\infty$ we get that $\delta = 0$. Similarly, $\liminf_{\ell\to 0}\int\hat\gamma_\ell\,\mathrm{d}p = \int\hat\gamma\,\mathrm{d}p$.

(iii). This is immediate from (ii).

(iv). We compute the difference

$$\begin{aligned}\left|\varepsilon^{\gamma_\ell}(p) - p^2\right| &= \frac{2}{(2\pi)^{3/2}}\left|\int\left(\hat V_\ell(p-q) - \hat V_\ell(0)\right)\hat\gamma_\ell(q)\,\mathrm{d}q\right|\\ &\leq \frac{2}{(2\pi)^3}\iint\left|V_\ell(x)\left(e^{-i(p-q)x} - 1\right)\right|\hat\gamma_\ell(q)\,\mathrm{d}q\,\mathrm{d}x\\ &\leq \frac{2}{(2\pi)^3}\|V_\ell\|_{L^1}\int\hat\gamma_\ell(q)\sup_{|x|\leq\ell}\left|e^{-i(p-q)x} - 1\right|\mathrm{d}q\\ &\leq C\|V_\ell\|_{L^1}\left(\left\|\hat\gamma_\ell|\cdot|^b\right\|_{L^1} + |p|^b\|\hat\gamma_\ell\|_{L^1}\right)\ell^b\end{aligned}$$

since we may bound $\left|e^{it} - 1\right| \leq C|t|^b$ and $|p-q|^b \leq (2\max\{|p|,|q|\})^b \leq 2^b(|p|^b + |q|^b)$. All these L^1-norms are uniformly bounded by (A4) and lemma 6.14. We conclude that $\varepsilon^{\gamma_\ell}$ converges to p^2 pointwise as desired.

(v). The Euler-Lagrange equation for γ_ℓ reads

$$\hat\gamma_\ell = \frac{1}{2} - \frac{\varepsilon^{\gamma_\ell} - \tilde\mu^{\gamma_\ell}}{2K_{T,\mu}^{\gamma_\ell,\Delta_\ell}} = \frac{1}{2} - \frac{\varepsilon^{\gamma_\ell} - \tilde\mu^{\gamma_\ell}}{2K_{T,\tilde\mu^{\gamma_\ell}}^{0,\Delta_\ell}}.$$

Now, we have just shown that the right-hand-side converges pointwise to

$$\tilde\gamma = \frac{1}{2} - \frac{p^2 - \tilde\mu^\gamma}{2K_{T,\tilde\mu^\gamma}^{0,\Delta}}.$$

The weak and pointwise limits coincide, however. This follows by the inequalities $0 \leq \hat\gamma_\ell \leq 1$ and Fatou's lemma [22, Thm. 9.11]. For any set A of finite measure we have the inequalities

$$\begin{aligned}\int_A\tilde\gamma\,\mathrm{d}p &= \int_A\liminf_{\ell\to 0}\hat\gamma_\ell\,\mathrm{d}p \leq \liminf_{\ell\to 0}\int_A\hat\gamma_\ell\,\mathrm{d}p = \int_A\hat\gamma\,\mathrm{d}p,\\ \int_A 1 - \tilde\gamma\,\mathrm{d}p &= \int_A\liminf_{\ell\to 0}1 - \hat\gamma_\ell\,\mathrm{d}p \leq \liminf_{\ell\to 0}\int_A 1 - \hat\gamma_\ell\,\mathrm{d}p = \int_A 1 - \hat\gamma\,\mathrm{d}p.\end{aligned}$$

Thus $\hat\gamma = \tilde\gamma$ satisfies equation (6.2).

(vi). The above arguments show that the integrand converges pointwise to the desired. We rewrite the integrand as a sum of two (types of) terms. One where we can use dominated convergence to get the desired, and one where we use an argument similar to the one we used in (ii).

Note that the Euler-Lagrange equation for γ_ℓ can be rewritten as

$$\hat\gamma_\ell = \frac{1}{1 + \exp\left(\frac{E_\mu^{\gamma_\ell,\Delta_\ell}}{T}\right)} + \frac{E_\mu^{\gamma_\ell,\Delta_\ell} - (\varepsilon^{\gamma_\ell} - \tilde\mu^{\gamma_\ell})}{2K_{T,\mu}^{\gamma_\ell,\Delta_\ell}}.$$

Thus, introducing $\xi(x) = \frac{x}{e^x - 1}$ we may write

$$\frac{1}{K_{T,\mu}^{\gamma_\ell,\Delta_\ell}} - \frac{1}{p^2} = \frac{\varepsilon^{\gamma_\ell} - \tilde{\mu}^{\gamma_\ell} - E_\mu^{\gamma_\ell,\Delta_\ell}}{K_{T,\mu}^{\gamma_\ell,\Delta_\ell} p^2} + \frac{p^2 - (\varepsilon^{\gamma_\ell} - \tilde{\mu}^{\gamma_\ell}) - 2T\xi\left(\frac{E_\mu^{\gamma_\ell,\Delta_\ell}}{T}\right)}{K_{T,\mu}^{\gamma_\ell,\Delta_\ell} p^2}$$

$$= -2\frac{\hat{\gamma}_\ell}{p^2} + \frac{2}{p^2}\frac{1}{1 + \exp\left(\frac{E_\mu^{\gamma_\ell,\Delta_\ell}}{T}\right)} + \frac{p^2 - (\varepsilon^{\gamma_\ell} - \tilde{\mu}^{\gamma_\ell}) - 2T\xi\left(\frac{E_\mu^{\gamma_\ell,\Delta_\ell}}{T}\right)}{K_{T,\mu}^{\gamma_\ell,\Delta_\ell} p^2}. \tag{6.4}$$

As before, we have $E_\mu^{\gamma_\ell,\Delta_\ell} \geq p^2 - \mu$ and so the middle term can be bounded by

$$\frac{2}{p^2}\frac{1}{1 + \exp\left(\frac{E_\mu^{\gamma_\ell,\Delta_\ell}}{T}\right)} \leq \frac{2}{p^2}\frac{1}{1 + \exp\left(\frac{p^2-\mu}{T}\right)} \in L^1.$$

Also,

$$\kappa(x) := \begin{cases} \frac{x}{\tanh x} & \text{if } x > 0, \\ 1 & \text{if } x \leq 0 \end{cases}$$

is increasing. Thus $K_{T,\mu}^{\gamma_\ell,\Delta_\ell} \geq 2T\kappa\left(\frac{p^2-\mu}{2T}\right)$. Moreover, $\xi(x) \leq 1$ for $x \geq 0$. Using the bound from (iv) we thus see that the latter term in equation (6.4) above can be dominated by an integrable function. Hence dominated convergence takes care of the middle and last terms. An argument similarly as in (ii) takes care of the $\hat{\gamma}_\ell/p^2$-term.

We proved above that γ also satisfies the Euler-Lagrange equation, and so equation (6.4) hold also for $(\gamma, \tilde{\mu}^\gamma, \Delta)$ instead. This concludes the proof. □

Now we are ready to prove our theorem

Proof of theorem 6.7. The convergence of $\gamma_\ell, \alpha_\ell, \Delta_\ell$ and that the limit points satisfy the appropriate Euler-Lagrange equations follow from lemma 6.15 for a subsequence. We now show, that the limit points of $\tilde{\mu}^{\gamma_\ell}$ and $|\Delta_\ell|$ and thus also that of $\hat{\gamma}_\ell$ are unique.

A pair of limit points $(\tilde{\mu}^\gamma, \Delta)$ satisfies the equations

$$F(\tilde{\mu}^\gamma, \Delta) = 0, \qquad G(\tilde{\mu}^\gamma, \Delta) = 0$$

where

$$F(\nu, \Delta) = \nu - \mu + \frac{\mathcal{V}}{(2\pi)^{3/2}} \int 1 - \frac{p^2 - \nu}{K_{T,\nu}^{0,\Delta}} \, \mathrm{d}p, \qquad G(\nu, \Delta) = \frac{1}{4\pi a} + \frac{1}{(2\pi)^3} \int \frac{1}{K_{T,\nu}^{0,\Delta}} - \frac{1}{p^2} \, \mathrm{d}p.$$

One computes that

$$\partial_\nu F > 0, \qquad \partial_\Delta F > 0, \qquad \partial_\nu G > 0, \qquad \partial_\Delta G < 0.$$

This proves uniqueness for $\Delta \neq 0$. For $\Delta = 0$ we have

$$F(\nu, 0) = \nu - \mu + \frac{\mathcal{V}}{(2\pi)^{3/2}} \int 1 - \tanh\left(\frac{p^2 - \nu}{2T}\right) \mathrm{d}p = \nu - \mu + \frac{2\mathcal{V}}{(2\pi)^{3/2}} \int \frac{1}{1 + \exp\left(\frac{p^2-\nu}{T}\right)} \, \mathrm{d}p.$$

In the proof of proposition 6.9 we saw that this was strictly increasing in ν, and so there is at most one solution to the equation $F(\nu, 0) = 0$.

It remains to consider a sequence of non-vanishing Δ_ℓ. We follow a method similar to section 4.3. For the minimisers we have the BCS gap equation

$$(K_{T,\mu}^{\gamma_\ell,\Delta_\ell} + V_\ell)\alpha_\ell = 0.$$

Since $\Delta_\ell \not\equiv 0$ we have that $\alpha_\ell \not\equiv 0$. By the Birman-Schwinger principle, $K_{T,\mu}^{\gamma_\ell,\Delta_\ell} + V_\ell$ has an eigenvalue 0 if and only if

$$V_\ell^{1/2} \frac{1}{K_{T,\mu}^{\gamma_\ell,\Delta_\ell}} |V_\ell|^{1/2}$$

has an eigenvalue -1. Decompose this as

$$V_\ell^{1/2} \frac{1}{K_{T,\mu}^{\gamma_\ell,\Delta_\ell}} |V_\ell|^{1/2} = V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2} + \tilde{m}_\mu^{\gamma_\ell,\Delta_\ell}(T) \left| V_\ell^{1/2} \right\rangle \left\langle |V_\ell|^{1/2} \right| + A_{\mu,T,\ell}$$

where $A_{\mu,T,\ell}$ is defined so that the above holds. Then its kernel is

$$A_{\mu,T,\ell}(x,y) = \frac{1}{(2\pi)^3} V_\ell(x)^{1/2} |V_\ell(y)|^{1/2} \int \left(\frac{1}{K_{T,\mu}^{\gamma_\ell,\Delta_\ell}} - \frac{1}{p^2} \right) \left(e^{-ip(y-x)} - 1 \right) \, \mathrm{d}p.$$

For the sake of simplifying notation, we introduce $Q_\ell := V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2}$. Then by (A9) we have $1 + Q_\ell$ invertible and so

$$1 + V_\ell^{1/2} \frac{1}{K_{T,\mu}^{\gamma_\ell,\Delta_\ell}} |V_\ell|^{1/2} = (1 + Q_\ell) \left(1 + \frac{1}{1 + Q_\ell} \left(\tilde{m}_\mu^{\gamma_\ell,\Delta_\ell}(T) \left| V_\ell^{1/2} \right\rangle \left\langle |V_\ell|^{1/2} \right| + A_{\mu,T,\ell} \right) \right).$$

Thus the latter operator has an eigenvalue of -1.

Claim 6.16. We have

$$\lim_{\ell \to 0} \left\| \frac{1}{1 + V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2}} A_{\mu,T,\ell} \right\| = 0.$$

We prove this below. This implies that $1 + \frac{1}{1+Q_\ell} A_{\mu,T,\ell}$ is invertible for small enough ℓ. Thus similarly as above

$$\tilde{m}_\mu^{\gamma_\ell,\Delta_\ell}(T) \left(1 + \frac{1}{1 + Q_\ell} A_{\mu,T,\ell} \right)^{-1} \frac{1}{1 + Q_\ell} \left| V_\ell^{1/2} \right\rangle \left\langle |V_\ell|^{1/2} \right|$$

has an eigenvalue of -1. This operator has 1-dimensional range, hence its trace is this eigenvalue, i.e.

$$-1 = \tilde{m}_\mu^{\gamma_\ell,\Delta_\ell}(T) \left\langle |V_\ell|^{1/2} \middle| \left(1 + \frac{1}{1 + Q_\ell} A_{\mu,T,\ell} \right)^{-1} \frac{1}{1 + Q_\ell} \middle| V_\ell^{1/2} \right\rangle.$$

For any operator S we have

$$\frac{1}{1 + S} = \sum_{k=0}^{\infty} (-S)^k = 1 - S \sum_{k=0}^{\infty} (-S)^k = 1 - S \frac{1}{1 + S}.$$

Using this with $S = \frac{1}{1+Q_\ell} A_{\mu,T,\ell}$ and recalling the definition of the scattering length we get

$$4\pi a(V_\ell) + \frac{1}{\tilde{m}_\mu^{\gamma_\ell,\Delta_\ell}(T)} = \left\langle |V_\ell|^{1/2} \middle| \frac{1}{1 + Q_\ell} A_{\mu,T,\ell} \left(1 + \frac{1}{1 + Q_\ell} A_{\mu,T,\ell} \right)^{-1} \frac{1}{1 + Q_\ell} \middle| V_\ell^{1/2} \right\rangle. \tag{6.5}$$

We now study the asymptotics of this to conclude the theorem.

Claim 6.17. The right hand side of equation (6.5) above vanishes as $\ell \to 0$.

We prove this below. It follows that

$$\lim_{\ell\to 0} \tilde{m}_\mu^{\gamma_\ell,\Delta_\ell}(T) = -\lim_{\ell\to 0} \frac{1}{4\pi a(V_\ell)} = \frac{-1}{4\pi a}.$$

However, by lemma 6.15(vi) above we have a subsequence with

$$\lim_{\ell\to 0} \tilde{m}_\mu^{\gamma_\ell,\Delta_\ell}(T) = \frac{1}{(2\pi)^3} \int \frac{1}{K_{T,\tilde{\mu}^\gamma}^{0,\Delta}} - \frac{1}{p^2}\, \mathrm{d}p.$$

Thus we get the desired for a subsequence. The uniqueness of the limit points finishes the proof. This is the following argument.

We may use lemma 6.15 and the argument above on any subsequence of the full sequence and so get a subsubsequence which converges as desired. The uniqueness of the limit points gives that any such subsubsequence converges to the same limit. This shows that the full sequence converges as desired. □

We now give the proofs of claims 6.16 and 6.17.

Proof of claim 6.16. Decompose $\frac{1}{1+Q_\ell}$ as

$$\frac{1}{1+Q_\ell} = \frac{1}{e_\ell} P_\ell + \frac{1}{1+Q_\ell}(1 - P_\ell)$$

where we used that ϕ_ℓ is an eigenvector of $1 + Q_\ell$ with eigenvalue e_ℓ. By (A9) the second summand is uniformly bounded. For $A_{\mu,T,\ell}$ we have that its kernel is bounded by

$$|A_{\mu,T,\ell}(x,y)| \leq C|V_\ell(x)|^{1/2}|V_\ell(y)|^{1/2} \int \left| \frac{1}{K_{T,\mu}^{\gamma_\ell,\Delta_\ell}} - \frac{1}{p^2} \right| (|x-y||p|)^q\, \mathrm{d}p$$

for any $0 \leq q \leq 1$. Similarly to the proof of lemma 6.15 (vi) we may prove that the integral is uniformly bounded in ℓ for any $q < 1$. So fix some $1/2 < q < 1$. Thus we get that

$$\left\|A_{\mu,T,\ell}\right\|_2^2 = \iint |A_{\mu,T,\ell}(x,y)|^2\, \mathrm{d}x\, \mathrm{d}y \leq C \iint |V_\ell(x)||V_\ell(y)||x-y|^{2q}\, \mathrm{d}x\, \mathrm{d}y \leq C\ell^{2q} \left\|V_\ell\right\|_{L^1}^2.$$

Thus

$$\left\| \frac{1}{1+Q_\ell}(1-P_\ell)A_{\mu,T,\ell} \right\| = O(\ell^q).$$

Now, it remains to show that $\frac{1}{e_\ell} P_\ell A_{\mu,T,\ell}$ vanishes. We have that

$$\left\|P_\ell A_{\mu,T,\ell}\right\| = \left\| \frac{|\phi_\ell\rangle \langle J_\ell \phi_\ell|}{\langle J_\ell\phi_\ell|\phi_\ell\rangle} A_{\mu,T,\ell} \right\| = \frac{\left\| A_{\mu,T,\ell}^\dagger |J_\ell\phi_\ell\rangle \langle \phi_\ell| \right\|}{|\langle J_\ell\phi_\ell|\phi_\ell\rangle|} = \frac{\left\| A_{\mu,T,\ell}^\dagger J_\ell\phi_\ell \right\|_{L^2}}{|\langle J_\ell\phi_\ell|\phi_\ell\rangle|}.$$

For the function $A_{\mu,T,\ell}^\dagger J_\ell\phi_\ell$ we have

$$\begin{aligned} \left| \left(A_{\mu,T,\ell}^\dagger J_\ell\phi_\ell\right)(y) \right| &\leq C|V_\ell(y)|^{1/2} \iint \left| \frac{1}{K_{T,\mu}^{\gamma_\ell,\Delta_\ell}} - \frac{1}{p^2} \right| |V_\ell(x)|^{1/2} (|x-y||p|)^q\, \mathrm{d}p |\phi_\ell(x)|\, \mathrm{d}x \\ &\leq C|V_\ell(y)|^{1/2} \ell^q \int |V_\ell(x)|^{1/2} |\phi_\ell(x)|\, \mathrm{d}x \end{aligned}$$

where, similarly as above, the p-integral is bounded uniformly in ℓ. Thus we get that

$$\left\|P_\ell A_{\mu,T,\ell}\right\| \leq C \left\|V_\ell\right\|_{L^1}^{1/2} \ell^q \frac{\left|\left\langle |V_\ell|^{1/2} \middle| |\phi_\ell| \right\rangle\right|}{|\langle J_\ell \phi_\ell | \phi_\ell \rangle|} = O\left(\ell^{q+1/2}\right)$$

by assumption (A10). Since $|e_\ell| > c\ell$ we thus get that

$$\left\| \frac{1}{1 + V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2}} A_{\mu,T,\ell} \right\| = O\left(\ell^{q-1/2}\right).$$

□

Proof of claim 6.17. We again decompose

$$\frac{1}{1+Q_\ell} = \frac{1}{e_\ell} P_\ell + \frac{1}{1+Q_\ell}(1 - P_\ell)$$

For the second half of the right-hand-side of equation (6.5) we thus have

$$\left(1 + \frac{1}{1+Q_\ell} A_{\mu,T,\ell}\right)^{-1} \frac{1}{1+Q_\ell} \left| |V_\ell|^{1/2} \right\rangle = \left(1 + \frac{1}{1+Q_\ell} A_{\mu,T,\ell}\right)^{-1} \left(\frac{1}{e_\ell} P_\ell + \frac{1}{1+Q_\ell}(1 - P_\ell)\right) \left| |V_\ell|^{1/2} \right\rangle.$$

This first factor is bounded and $\frac{1}{1+Q_\ell}(1 - P_\ell)$ is as well by (A9) thus we bound

$$\left\|P_\ell |V_\ell|^{1/2}\right\|_{L^2} = \frac{\left\| |\phi_\ell\rangle \left\langle J_\ell \phi_\ell \middle| |V_\ell|^{1/2} \right\rangle \right\|_{L^2}}{|\langle J_\ell \phi_\ell | \phi_\ell \rangle|} = \frac{\left|\left\langle |V_\ell|^{1/2} \middle| J_\ell \phi_\ell \right\rangle\right|}{|\langle J_\ell \phi_\ell | \phi_\ell \rangle|} = O\left(\ell^{1/2}\right)$$

by (A10). Thus we get that

$$\left\| \left(1 + \frac{1}{1+Q_\ell} A_{\mu,T,\ell}\right)^{-1} \frac{1}{1+Q_\ell} \left| |V_\ell|^{1/2} \right\rangle \right\|_{L^2} = O\left(\ell^{-1/2}\right).$$

For the first half we have

$$\left\| \left\langle |V_\ell|^{1/2} \right| \frac{1}{1+Q_\ell} A_{\mu,T,\ell} \right\| = \left\| A_{\mu,T,\ell}^\dagger \frac{1}{1+Q_\ell^\dagger} |V_\ell|^{1/2} \right\|_{L^2}.$$

We now decompose

$$\frac{1}{1+Q_\ell^\dagger} = \frac{1}{e_\ell} P_\ell^\dagger + (1 - P_\ell^\dagger) \frac{1}{1+Q_\ell^\dagger} = \frac{1}{e_\ell} P_\ell^\dagger P_\ell^\dagger + (1 - P_\ell^\dagger) \frac{1}{1+Q_\ell^\dagger},$$

since $P_\ell^2 = P_\ell$. Thus we should bound the L^2-norm of

$$A_{\mu,T,\ell}^\dagger \frac{1}{1+Q_\ell^\dagger} |V_\ell|^{1/2} = \frac{1}{e_\ell} A_{\mu,T,\ell}^\dagger P_\ell^\dagger P_\ell^\dagger |V_\ell|^{1/2} + A_{\mu,T,\ell}^\dagger (1 - P_\ell^\dagger) \frac{1}{1+Q_\ell^\dagger} |V_\ell|^{1/2}.$$

The first term is bounded by

$$\frac{1}{e_\ell} \left\|P_\ell A_{\mu,T,\ell}\right\| \left\|P_\ell^\dagger |V_\ell|^{1/2}\right\|_{L^2} = O(\ell^q)$$

by the bounds $\left\|P_\ell A_{\mu,T,\ell}\right\| = O(\ell^{q+1/2})$ and $\left\|P_\ell V_\ell^{1/2}\right\|_{L^2} = O(\ell^{1/2})$. Also, by (A9), (A4) and the bound $\left\|A_{\mu,T,\ell}\right\| = O(\ell^q)$, we get that the second term is $O(\ell^q)$. In total we thus get the bound

$$\left\| A_{\mu,T,\ell}^\dagger \frac{1}{1+Q_\ell^\dagger} |V_\ell|^{1/2} \right\|_{L^2} = O(\ell^q).$$

We conclude that the right hand side of equation (6.5) is $O\left(\ell^{q-1/2}\right)$ in the limit $\ell \to 0$. Hence it vanishes as desired. □

6.2.2 The Critical Temperature

Now, we turn our attention to theorem 6.12. We have the following.

Lemma 6.18. *Let $\mu > 0$ and $T < T_c$. Let $\Gamma_\ell^0 = (\gamma_\ell^0, 0)$ be a sequence of normal states. Then*

$$\liminf_{\ell \to 0} \tilde{m}_\mu^{\gamma_\ell^0, 0}(T) > \frac{-1}{4\pi a}.$$

Proof. This is similar to lemma 6.15. We have

$$\lim_{\ell \to 0} \tilde{m}_\mu^{\gamma_\ell^0, 0}(T) = \tilde{m}_{\tilde{\mu}^\gamma}^{0,0}(T)$$

where $\tilde{\mu}^\gamma$ and γ are as above. By the proof of proposition 6.9 we have that $\tilde{m}_{\tilde{\mu}^\gamma}^{0,0}(T)$ is decreasing in T. At $T = T_c$ it equals $\frac{-1}{4\pi a}$. We conclude the desired. □

Lemma 6.19. *Let $\Gamma_\ell^0 = (\gamma_\ell^0, 0)$ be a sequence of normal states. Suppose that $\lim_{\ell \to 0} \tilde{m}_\mu^{\gamma_\ell^0, 0}(T) > \frac{-1}{4\pi a}$. Then, for small enough ℓ the operator $K_{T,\mu}^{\gamma_\ell^0, 0} + V_\ell$ has a negative eigenvalue.*

Together with theorem 6.4 this means that the system is in a superconducting state.

Proof. We use the Birman-Schwinger principle to relate the existence of a negative eigenvalue of $K_{T,\mu}^{\gamma_\ell^0, 0} + V_\ell$ to eigenvalues of some Birman-Schwinger operator, which we then decompose as previously. Factoring out terms of the form $1 + S$ as before, we relate this to an inequality between $\tilde{m}_\mu^{\gamma_\ell^0, 0}(T)$ and the scattering length.

By the Birman-Schwinger principle $K_{T,\mu}^{\gamma_\ell^0, 0} + V_\ell$ has an eigenvalue of $e \leq 0$ if and only if the associated Birman-Schwinger operator $V_\ell^{1/2} \frac{1}{K_{T,\mu}^{\gamma_\ell^0, 0} - e} |V_\ell|^{1/2}$ has an eigenvalue of -1. We decompose this operator as

$$V_\ell^{1/2} \frac{1}{K_{T,\mu}^{\gamma_\ell^0, 0} - e} |V_\ell|^{1/2} = V_\ell^{1/2} \frac{1}{p^2} |V_\ell|^{1/2} + \tilde{m}_{\mu,e}^{\gamma_\ell^0, 0}(T) \left| V_\ell^{1/2} \right\rangle \left\langle |V_\ell|^{1/2} \right| + A_{\mu,T,\ell,e}$$

where

$$\tilde{m}_{\mu,e}^{\gamma_\ell^0, 0}(T) = \frac{1}{(2\pi)^3} \int \frac{1}{K_{T,\mu}^{\gamma_\ell^0, 0} - e} - \frac{1}{p^2} \, \mathrm{d}p.$$

Now, $\left\| A_{\mu,T,\ell,e} \right\| = O(\ell^q)$ as in claim 6.16. We need only check that

$$\int |p|^q \left| \frac{1}{K_{T,\mu}^{\gamma_\ell^0, 0} - e} - \frac{1}{p^2} \right| \mathrm{d}p \leq \int |p|^q \left| \frac{1}{K_{T,\mu}^{\gamma_\ell^0, 0} - e} - \frac{1}{K_{T,\mu}^{\gamma_\ell^0, 0}} \right| \mathrm{d}p + \int |p|^q \left| \frac{1}{K_{T,\mu}^{\gamma_\ell^0, 0}} - \frac{1}{p^2} \right| \mathrm{d}p$$

is bounded uniformly in ℓ. The second term is fine. For the first we have

$$\left| \frac{1}{K_{T,\mu}^{\gamma_\ell^0, 0} - e} - \frac{1}{K_{T,\mu}^{\gamma_\ell^0, 0}} \right| = \frac{|e|}{K_{T,\mu}^{\gamma_\ell^0, 0} \left(K_{T,\mu}^{\gamma_\ell^0, 0} - e \right)} \leq \frac{|e|}{(2T)^2} \frac{1}{\kappa\left(\frac{p^2 - \mu}{2T}\right) \left(\kappa\left(\frac{p^2 - \mu}{2T}\right) - \frac{e}{2T} \right)}.$$

Since this is integrable we get that $A_{\mu,T,\ell,e}$ is bounded as desired. As in the proof of theorem 6.7 we have that

$$1 + V_\ell^{1/2}\frac{1}{p^2}|V_\ell|^{1/2} + A_{\mu,T,\ell,e} = \left(1 + V_\ell^{1/2}\frac{1}{p^2}|V_\ell|^{1/2}\right)\left(1 + \frac{1}{1 + V_\ell^{1/2}\frac{1}{p^2}|V_\ell|^{1/2}}A_{\mu,T,\ell,e}\right)$$

is invertible for small ℓ. This follows from a suitably changed reformulation of claim 6.16, the proof of which follows by the bound $\left\|A_{\mu,T,\ell,e}\right\| = O(\ell^q)$ above and the proof of claim 6.16. Thus, similarly as before we get that $K_{T,\mu}^{\gamma_\ell^0,0} + V_\ell$ has an eigenvalue $e < 0$ if and only if

$$4\pi\tilde{a}_{\ell,e} := \left\langle |V_\ell|^{1/2}\middle|\frac{1}{1 + V_\ell^{1/2}\frac{1}{p^2}|V_\ell|^{1/2} + A_{\mu,T,\ell,e}}\middle|V_\ell^{1/2}\right\rangle = \frac{-1}{\tilde{m}_{\mu,e}^{\gamma_\ell^0,0}(T)}.$$

We claim that for small enough ℓ this equation has a solution $e < 0$.

First, we show that $\lim_\ell 4\pi\tilde{a}_{\ell,e} = 4\pi a$. This is similar to the discussion preceding claim 6.17. We compute

$$\begin{aligned}
4\pi a(V_\ell) - 4\pi\tilde{a}_{\ell,e} &= \left\langle |V_\ell|^{1/2}\middle|\frac{1}{1+Q_\ell} - \frac{1}{1+Q_\ell + A_{\mu,T,\ell,e}}\middle|V_\ell^{1/2}\right\rangle \\
&= \left\langle |V_\ell|^{1/2}\middle|\left(1 - \left(1 + \frac{1}{1+Q_\ell}A_{\mu,T,\ell,e}\right)^{-1}\right)\frac{1}{1+Q_\ell}\middle|V_\ell^{1/2}\right\rangle \\
&= \left\langle |V_\ell|^{1/2}\middle|\frac{1}{1+Q_\ell}A_{\mu,T,\ell,e}\left(1 + \frac{1}{1+Q_\ell}A_{\mu,T,\ell,e}\right)^{-1}\frac{1}{1+Q_\ell}\middle|V_\ell^{1/2}\right\rangle
\end{aligned}$$

where we again used that $\frac{1}{1+S} = 1 - S\frac{1}{1+S}$ for any operator S. This vanishes similarly as in claim 6.17. Thus, $\lim \tilde{4\pi a}_{\ell,e} = 4\pi a$ as desired.

Now, $\tilde{m}_{\mu,e}^{\gamma_\ell^0,0}$ is easily seen to be increasing in e and $(-\infty, 0] \ni e \mapsto \tilde{m}_{\mu,e}^{\gamma_\ell^0,0}(T)$ has image $\left(-\infty, \tilde{m}_\mu^{\gamma_\ell^0,0}\right]$. By assumption $\lim_\ell \tilde{m}_\mu^{\gamma_\ell^0,0}(T) > \frac{-1}{4\pi a}$. Now, $\tilde{a}_{\ell,e}$ is continuous in e. To see this, note that the resolvent $\frac{1}{K_{T,\mu}^{\gamma_\ell^0,0} - e}$ is continuous in e and so $\tilde{m}_{\mu,e}^{\gamma_\ell^0,0}(T)$ is. Thus we conclude the existence of a solution of the above equation for ℓ sufficiently small and thus the existence of a negative eigenvalue for $K_{T,\mu}^{\gamma_\ell^0,0} + V_\ell$. □

Now, we may combine this to prove theorem 6.12.

Proof of theorem 6.12. Part (i) is clear from lemmas 6.18 and 6.19 and theorem 6.4.

For part (ii) suppose for contradiction that for some sequence of ℓ's going to zero, we have that for the minimisers $\alpha_\ell \not\equiv 0$. Then by theorem 6.7 the limit $(\gamma, \tilde{\mu}^\gamma, \Delta)$ satisfies the equations

$$\tilde{\mu}^\gamma = \mu - \frac{\mathcal{V}}{(2\pi)^{3/2}}\int 1 - \frac{p^2 - \tilde{\mu}^\gamma}{K_{T,\tilde{\mu}^\gamma}^{0,\Delta}}\,\mathrm{d}p, \qquad \text{and} \qquad \frac{-1}{4\pi a} = \frac{1}{(2\pi)^{3/2}}\int \frac{1}{K_{T,\tilde{\mu}^\gamma}^{0,\Delta}} - \frac{1}{p^2}\,\mathrm{d}p.$$

We show that these equation does not have a solution for $T > T_c$.

Seeing the first equation as defining a function $\tilde{\mu}^\gamma(\Delta)$ indirectly, we saw in the proof of theorem 6.7 that this function is decreasing in Δ. Moreover, the right-hand-side of the second equation is increasing in $\tilde{\mu}^\gamma$ and decreasing in Δ. Thus, decreasing Δ to 0 we get

$$\frac{-1}{4\pi a} \le \frac{1}{(2\pi)^3}\int \frac{\tanh\frac{p^2 - \tilde{\mu}_0}{2T}}{p^2 - \tilde{\mu}_0^\gamma} - \frac{1}{p^2}\,\mathrm{d}p. \tag{6.6}$$

where
$$\tilde{\mu}_0^\gamma = \mu - \frac{\mathcal{V}}{(2\pi)^{3/2}} \int 1 - \frac{p^2 - \tilde{\mu}_0^\gamma}{K_{T,\tilde{\mu}_0}^{0,0}} \,\mathrm{d}p = \mu - \frac{2\mathcal{V}}{(2\pi)^{3/2}} \int \frac{1}{1 + \exp\left(\frac{p^2 - \tilde{\mu}_0^\gamma}{T}\right)} \,\mathrm{d}p$$
is the value of the function $\tilde{\mu}^\gamma(\Delta)$ at $\Delta = 0$. In the proof of proposition 6.9 we saw that the right-hand-side of equation (6.6) is decreasing in T and equal to $\frac{-1}{4\pi a}$ at $T = T_c$. We conclude that $T \leq T_c$. Contradiction. Hence, for sufficiently small ℓ we have that the system is not superconducting. □

7 More General Interaction

In this section we explore to what extent we may prove earlier results for a different class of interaction operators. We will consider the setting where the potential operators no longer is a multiplication operator.

In BCS's original article [4] they consider an interaction term given by an operator V with integral kernel $V(x, y) = -V_0\phi(x)\phi(y)$, where $\hat{\phi}(p) = 1_{\{|p^2-\mu|<h\}}(p)$ and $V_0 > 0$ is some positive constant, see [16]. We change the interaction term of the BCS functional equation (2.2) from $\int V|\alpha|^2 \,\mathrm{d}x$ to the more general $\langle\alpha|V|\alpha\rangle$, which is meaningful for such more general operators V. Thus the BCS functional is
$$\mathcal{F}(\Gamma) := \int (p^2 - \mu)\gamma(p) \,\mathrm{d}p + \langle\alpha|V|\alpha\rangle - TS(\Gamma).$$
This section is new and based my own work, partially inspired by [19], meaning that the technical constructions in some the proofs in this section are from there.

7.1 The Potentials Under Study

We describe the set of potentials, that we will investigate. These will be some set $\mathcal{V}$ of operators. In this section we define this set $\mathcal{V}$ and describe some of its elements.

The set $\mathcal{V}$ is a subset of the quadratic forms on H^1. Any quadratic form on H^1 is a bounded operator $H^1 \to H^{-1}$ and so we have the topology induced by the norm
$$\|V\|_{B(H^1,H^{-1})} = \sup_{\|\alpha\|_{H^1}\leq 1} \|V\alpha\|_{H^{-1}} = \sup_{\|\alpha\|_{H^1},\|\beta\|_{H^1}\leq 1} |\langle\beta|V|\alpha\rangle| \,.$$
Note that if $V_n \to V$ in this norm then $\langle\alpha|V_n - V|\alpha\rangle \to 0$ uniformly in (norm-)bounded $\alpha \in H^1$. This is the main aspect of this topology, that we will use. (By a simple polarisation argument this is even equivalent to $V_n \to V$ in the norm above.)

We define the set $\mathcal{V}$ to be the closure (in this norm) of the set of selfadjoint reflection-symmetric operators (we define this below) with real-valued kernel $V(x, y) \in L^{6/5}(\mathbb{R}^3 \times \mathbb{R}^3) + L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, meaning that $V = v + w$ can be split with v having an $L^{6/5}$-kernel and w having an L^2-kernel, where both $v(x, y)$ and $w(x, y)$ are real-valued. That is,
$$\mathcal{V} = \overline{\left\{\text{Selfadjoint and reflection-symmetric } V \text{ with real-valued kernel } V(x,y) \in L^{6/5} + L^2\right\}}.$$
Now, to define what it means for an operator V to be reflection-symmetric.

Definition 7.1. Let u be a distribution and let T be an operator with form-domain $\mathcal{H} \subset L^2$. (Thus for some $\psi \in \mathcal{H}$ we may have that $T\psi \in \mathcal{H}^*$ is a proper distribution.) We say that

- $\psi \in C_0^\infty$ is reflection-anti-symmetric if $\psi(-x) = -\psi(x)$ for all x.

- u is reflection-symmetric if $\langle u, \psi \rangle = 0$ for all reflection-anti-symmetric $\psi \in C_0^\infty$.
- T is reflection-symmetric if $T\psi$ is reflection-symmetric for all reflection-symmetric $\psi \in \mathcal{H}$.

Note that the condition for a distribution to be reflection-symmetric is the same as $u(x) = u(-x)$ for a.e. x, when u is also a function. Also, the property of an operator being reflection-symmetric is closed in the topology above. Moreover, if an operator V has kernel and if, say $V(x,y) = V(-x,y)$ for all x,y or $V(x,y) = V(-x,-y)$ for all x,y, then V is reflection-symmetric. Additionally, for the kernel $V(x,y) = -V_0\phi(x)\phi(y)$ discussed above, we have that $\hat{\phi} \in L^2$ and so $V(x,y) \in L^2$. It is also reflection-symmetric, and so $V \in \mathcal{V}$.

We would ideally want, that $\mathcal{V}$ encompasses the multiplication operators studied in sections 2 to 6. (The results of this section of course applies to such multiplication operators. This is just the results in [19] referenced in those sections. Here we just consider the question, as to whether these operators are included here as well.) This is however not clear. We now prove that $\mathcal{V}$ indeed contains the multiplication operators $V \in L^{3/2}$. First, we give some motivation for why this is not so trivial.

A multiplication operator $V \in L^{3/2}$ does not have an integral kernel. Formally, its kernel is given by $V(x)\delta(x-y)$, which is of course not a function. We can, however, approximate the δ-function as $t^{-3}\chi((x-y)/t)$ for some small $t > 0$ and a compactly supported function χ. Such operators do converge to a multiplication operator, just not in a sense as strong as needed.

In order to state this result we first define the following. For any compactly supported positive χ with $\int \chi \, \mathrm{d}x = 1$ we define $\chi_t(x) := t^{-3}\chi(x/t)$.

Proposition 7.2. *Let $V \in L^{3/2}$ and define V_t for $t > 0$ by $V_t(x,y) = V(x)\chi_t(x-y)$. Then for every $\alpha, \beta \in H^1$ we have $\langle \beta | V_t | \alpha \rangle \to \langle \beta | V | \alpha \rangle$ as $t \to 0$.*

Proof. First, $\chi_t * \alpha \to \alpha$ in L^6 as $t \to 0$ by [8, Thm. 8.14]. Thus, an application of Hölder's and Sobolev's inequalities [19, Thm. 8.3] give the desired. ◻

In order to conclude that $V \in \mathcal{V}$ we need the convergence to be uniform in α and β. The problem is that the convergence is not uniform in α. The error in the dominated convergence part of the argument (which is how [8, Thm. 8.14] is proved) can only easily be bounded by something like $\varepsilon \left\| \nabla \alpha \right\|_{L^6}$ through a Taylor expansion. This is of course not uniformly bounded in $\alpha \in H^1$.

Note that the kernel $V_t(x,y) = V(x)\chi_t(x-y) \in L^{6/5} + L^2$. This follows from the general inclusion $L^p \subset L^q + L^r$ for any triple $q < p < r$, which can be seen by decomposing a function $f \in L^p$ as $f 1_{\{|f|>1\}} + f 1_{\{|f|\le 1\}} \in L^q + L^r$.

Refining this type of argument, we do however get some multiplication operators. If $V \in L^p$ for some larger p, then we could get the error term to be of the form $\varepsilon \left\| \nabla \alpha \right\|_{L^2}$, which would be fine. This is the extent of the following. The proof will however not go through such a Taylor expansion argument.

Proposition 7.3. *Let $V \in L^2 \cap L^3$ be real-valued and reflection-symmetric. Then $V \in \mathcal{V}$.*

The assumption $V \in L^2 \cap L^3$ is the weakest we can make for an argument like this to work. We need $V \in L^p$ for a small enough p such that the approximating kernels are in $L^{6/5} + L^2$. This means $p \le 2$. Also we need $V \in L^q$ for a large enough q such that Hölder's inequality will give the right powers for the α-norms. This means $q \ge 3$. The smallest set of the form $L^p \cap L^q$ with these restrictions is $L^2 \cap L^3$.

This (in some sense) weaker result is what we will use to show that $V \in L^{3/2}$ satisfies $V \in \mathcal{V}$.

Proof. Define V_t by $V_t(x,y) = V(x)\chi_t(x-y)$, where χ_t is as above for a specific choice of χ which we give below. Note that since $V \in L^2$ we have $V_t \in \mathcal{V}$. Then

$$|\langle \beta | V - V_t | \alpha \rangle| = \left| \int \overline{\beta(x)} V(x) \left(\chi_t * \alpha(x) - \alpha(x) \right) \, \mathrm{d}x \right| \le \left\| \beta \right\|_{L^6} \left\| V \right\|_{L^3} \left\| \chi_t * \alpha - \alpha \right\|_{L^2}.$$

Sobolev's inequality [19, Thm. 8.3] takes care of the first factor. For the last factor we have

$$\begin{aligned}\|\chi_t * \alpha - \alpha\|_{L^2}^2 &= \left\|(2\pi)^{3/2}\hat{\chi}_t\hat{\alpha} - \hat{\alpha}\right\|_{L^2}^2 \\ &= \int \left|\left((2\pi)^{3/2}\hat{\chi}_t(p) - 1\right)\hat{\alpha}(p)\right|^2 \frac{1+p^2}{1+p^2}\,\mathrm{d}p \\ &\leq \left\|\frac{(2\pi)^{3/2}\hat{\chi}_t - 1}{1+p^2}\right\|_{L^\infty}^2 \|\alpha\|_{H^1}^2 .\end{aligned}$$

We now want to show that the first factor vanishes as $t \longrightarrow 0$. For this we choose

$$\chi(x) = \frac{\frac{1}{|x|}1_{B(0,1)}(x)}{\int_{B(0,1)} \frac{1}{|y|}\,\mathrm{d}y} = \frac{1}{2\pi|x|}1_{B(0,1)}(x).$$

Then we compute

$$g(p) := \frac{(2\pi)^{3/2}\hat{\chi}_t(p) - 1}{1+p^2} = \frac{\frac{2}{(|p|t)^2}(1-\cos(|p|t)) - 1}{1+p^2}.$$

Let $\varepsilon > 0$. Since the numerator vanishes in $pt \longrightarrow 0$ we may find q_0 such that for $|p|t < q_0$ we have that $|g(p)| < \varepsilon$. The numerator also vanishes in $|p|t \longrightarrow \infty$ and so it is globally bounded by a constant C. Now find p_0 such that for $|p| > p_0$ we have $\frac{C}{1+p^2} < \varepsilon$. Then for $t < t_0 := \frac{q_0}{p_0}$ we have $|g(p)| < \varepsilon$ for all p. This shows that $\|g\|_{L^\infty} \longrightarrow 0$ as $t \longrightarrow 0$. Hence this shows the desired. □

Now, we are ready to show that $V \in L^{3/2}$ indeed satisfies $V \in \mathcal{V}$.

Proposition 7.4. *Let $V \in L^{3/2}$ be real-valued and reflection-symmetric. Then $V \in \mathcal{V}$.*

Proof. We may approximate V by $L^2 \cap L^3$ functions as follows. Define for $\delta > 0$ the functions

$$V^\delta(x) = \begin{cases} V(x) & \text{if } |V(x)| \leq \frac{1}{\delta}, \\ 0 & \text{if } |V(x)| > \frac{1}{\delta}. \end{cases}$$

Then $V^\delta \in L^{3/2} \cap L^\infty \subset L^2 \cap L^3$ so $V_\delta \in \mathcal{V}$ and by dominated convergence we have $V^\delta \longrightarrow V$ in $L^{3/2}$ as $\delta \longrightarrow 0$. We have for any $\alpha, \beta \in H^1$ that

$$\begin{aligned}\left|\left\langle\beta\middle|V - V^\delta\middle|\alpha\right\rangle\right| &\leq \int |\beta(x)||V(x) - V^\delta(x)||\alpha(x)|\,\mathrm{d}x \\ &\leq \left\|V - V^\delta\right\|_{L^{3/2}}\|\alpha\|_{L^6}\|\beta\|_{L^6} \leq C\left\|V - V^\delta\right\|_{L^{3/2}}\|\alpha\|_{H^1}\|\beta\|_{H^1}\end{aligned}$$

by Sobolev's inequality [19, Thm. 8.3]. This gives the desired. □

By a similar approximation argument, one may show that any (real-valued and reflection-symmetric) $V \in L^\infty_\varepsilon$ satisfies $V \in \mathcal{V}$. This is the other class of V's under study in [19]. Here L^∞_ε denotes the L^∞ functions vanishing weakly, in the sense that $\{|V| > \varepsilon\}$ has finite measure for any $\varepsilon > 0$.

7.2 Minimisers and the Linear Criterion

Now, we turn our attention to proving the results of section 3 for potentials $V \in \mathcal{V}$. The proof of theorem 3.1 (existence of minimisers) carries over without change once we prove that for any $V \in \mathcal{V}$

- If α_n converges weakly to $\alpha \in H^1$, then $\langle\alpha_n|V|\alpha_n\rangle \to \langle\alpha|V|\alpha\rangle$, and
- The operator $p^2 + V$ is bounded from below.

This is the extent of the following.

Proposition 7.5. *Let $V \in \mathcal{V}$. Suppose α_n converges weakly to α in H^1. Then $\langle\alpha_n|V|\alpha_n\rangle \to \langle\alpha|V|\alpha\rangle$.*

The general idea of this proof is from [19, Thm. 11.4]. We generalise this proof to our setting.

Proof. By a simple approximation argument exactly as the one we do below for v and w we may assume that $V = v + w$ with v having an $L^{6/5}$-kernel and w having an L^2-kernel.

By the uniform boundedness principle [8, Thm. 5.13] we have that $\|\alpha_n\|_{H^1} \le C$ is bounded independent of n. Define for all $\delta > 0$ the operators v^δ, w^δ to have integral kernels

$$v^\delta(x,y) = \begin{cases} v(x,y) & \text{if } |v(x,y)| \le 1/\delta, \\ 0 & \text{if } |v(x,y)| > 1/\delta. \end{cases}, \qquad w^\delta(x,y) = \begin{cases} w(x,y) & \text{if } |w(x,y)| \le 1/\delta, \\ 0 & \text{if } |w(x,y)| > 1/\delta. \end{cases}$$

respectively. Note that both of these kernels are bounded. Also, by dominated convergence we have $v^\delta(x,y) \to v(x,y)$ in $L^{6/5}$ and $w^\delta(x,y) \to w(x,y)$ in L^2. Then, by Sobolev's inequality [19, Thm. 8.3]

$$\left|\left\langle\alpha_n\middle|v - v^\delta\middle|\alpha_n\right\rangle\right| \le \|\alpha_n \otimes \alpha_n\|_{L^6(\mathbb{R}^3\times\mathbb{R}^3)} \left\|v - v^\delta\right\|_{L^{6/5}(\mathbb{R}^3\times\mathbb{R}^3)} \le C\,\|\alpha_n\|_{H^1}^2 \left\|v - v^\delta\right\|_{L^{6/5}} \le C_\delta$$

for a constant C_δ independent of n, satisfying $C_\delta \to 0$ as $\delta \to 0$. Also

$$\left|\left\langle\alpha_n\middle|w - w^\delta\middle|\alpha_n\right\rangle\right| \le \|\alpha_n\|_{L^2}^2 \left\|w - w^\delta\right\|_{L^2} \le \|\alpha_n\|_{H^1}^2 \left\|w - w^\delta\right\|_{L^2} \le C_\delta$$

for a (potentially different) constant C_δ also satisfying $C_\delta \to 0$ as $\delta \to 0$. Hence it suffices to prove that $\left\langle\alpha_n\middle|v^\delta\middle|\alpha_n\right\rangle \to \left\langle\alpha\middle|v^\delta\middle|\alpha\right\rangle$ and $\left\langle\alpha_n\middle|w^\delta\middle|\alpha_n\right\rangle \to \left\langle\alpha\middle|w^\delta\middle|\alpha\right\rangle$.

Define for all $\varepsilon > 0$ the sets

$$A_\varepsilon = \{(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3 : |v(x,y)| > \varepsilon\}, \qquad \text{and} \qquad B_\varepsilon = \{(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3 : |w(x,y)| > \varepsilon\}.$$

Then we have that $|A_\varepsilon|, |B_\varepsilon| < \infty$ and we may write

$$\begin{aligned} \left\langle\alpha_n\middle|v^\delta\middle|\alpha_n\right\rangle &= \left\langle\alpha_n\middle|v^\delta 1_{A_\varepsilon}\middle|\alpha_n\right\rangle + \left\langle\alpha_n\middle|v^\delta 1_{A_\varepsilon^c}\middle|\alpha_n\right\rangle, \\ \left\langle\alpha_n\middle|w^\delta\middle|\alpha_n\right\rangle &= \left\langle\alpha_n\middle|w^\delta 1_{B_\varepsilon}\middle|\alpha_n\right\rangle + \left\langle\alpha_n\middle|w^\delta 1_{B_\varepsilon^c}\middle|\alpha_n\right\rangle. \end{aligned}$$

For the second terms, we may by the same arguments as above bound

$$\begin{aligned} \left|\left\langle\alpha_n\middle|v^\delta 1_{A_\varepsilon^c}\middle|\alpha_n\right\rangle\right| &\le C\left\|1_{A_\varepsilon^c} v^\delta\right\|_{L^{6/5}} \le C\left\|1_{A_\varepsilon^c} v\right\|_{L^{6/5}}, \\ \left|\left\langle\alpha_n\middle|w^\delta 1_{B_\varepsilon^c}\middle|\alpha_n\right\rangle\right| &\le C\left\|1_{B_\varepsilon^c} w^\delta\right\|_{L^2} \le C\left\|1_{B_\varepsilon^c} w\right\|_{L^2}. \end{aligned}$$

Both of these converge to 0 as $\varepsilon \to 0$ by dominated convergence.

Now, for the first terms we have by [19, Thm. 8.6] that $\alpha_n \otimes \alpha_n \to \alpha \otimes \alpha$ strongly in $L^p(A_\varepsilon)$ for any $p < 3$. Hence fix such a $1 < p < 3$. Since $1_{A_\varepsilon} v^\delta(x,y) \in L^\infty(A_\varepsilon)$ we have that $1_{A_\varepsilon} v^\delta(x,y) \in L^{p'}(A_\varepsilon)$ for p' the dual exponent. We conclude that $\left\langle\alpha_n\middle|1_{A_\varepsilon} v^\delta\middle|\alpha_n\right\rangle$ converges to $\left\langle\alpha\middle|1_{A_\varepsilon} v^\delta\middle|\alpha\right\rangle$ as desired. The same argument applied to B_ε and w^δ gives that $\left\langle\alpha_n\middle|1_{B_\varepsilon} w^\delta\middle|\alpha_n\right\rangle$ converges to $\left\langle\alpha\middle|1_{B_\varepsilon} w^\delta\middle|\alpha\right\rangle$. Putting everything together we conclude the proposition. □

Proposition 7.6. *Let $V \in \mathcal{V}$. Then $p^2 + V$ is bounded from below.*

The general idea of this proof is from [19, Thm. 11.3]. We generalise this proof to our setting.

Proof. Again, we may by a simple approximation argument assume that $V = v + w$ with v having an $L^{6/5}$-kernel and w an L^2-kernel.

Define for $\lambda \in \mathbb{R}$ the function $h_\lambda = -(v-\lambda)_- = \min\{v-\lambda, 0\}$. Note that $h_\lambda \to 0$ pointwise for $\lambda \to -\infty$. Hence by dominated convergence we have $\|h_{\lambda_0}\|_{L^{6/5}} \le \frac{1}{2S_3}$ for some λ_0, denote this function by $h := h_{\lambda_0}$. Here S_3 is the constant from Sobolev's inequality [19, Thm. 8.3]. Thus, by a similar argument as above, using Sobolev's inequality [19, Thm. 8.3], we get that

$$|\langle\psi|h|\psi\rangle| \le \|h\|_{L^{6/5}} \|\psi\otimes\psi\|_{L^6} \le \frac{1}{2}\|\nabla\psi\|_{L^2}^2 = \frac{1}{2}\left\langle\psi\middle|p^2\middle|\psi\right\rangle$$

for any function ψ. Thus, for any ψ with $\|\psi\|_{L^2} = 1$ we have

$$\begin{aligned}\left\langle\psi\middle|p^2+V\middle|\psi\right\rangle &= \left\langle\psi\middle|p^2+(v-\lambda_0)+\lambda_0+w\middle|\psi\right\rangle \ge \left\langle\psi\middle|p^2+h\middle|\psi\right\rangle + \lambda_0 - \|w\|_{L^2}\\ &\ge \frac{1}{2}\left\langle\psi\middle|p^2\middle|\psi\right\rangle + \lambda_0 - \|w\|_{L^2} > -\infty.\end{aligned}$$

□

We conclude that the result of theorem 3.1 holds.

Corollary 7.7. *Let $V \in \mathcal{V}$ and let $\mu \in \mathbb{R}$. Then*

$$\mathcal{F}(\Gamma) = \int (p^2-\mu)\gamma(p)\,\mathrm{d}p + \langle\alpha|V|\alpha\rangle - TS(\Gamma)$$

is bounded from below and attains its minimum on

$$\mathcal{D} = \left\{\Gamma(p) = \begin{pmatrix}\gamma(p) & \hat\alpha(p)\\ \bar{\hat\alpha}(p) & 1-\gamma(-p)\end{pmatrix} : \alpha \in H^1(\mathbb{R}^3),\ \gamma \in L^1\left(\mathbb{R}^3, (1+p^2)\,\mathrm{d}p\right),\ 0 \le \Gamma \le 1\right\}.$$

Moreover, a minimising $\Gamma = (\gamma, \alpha)$ satisfies the BCS gap equation

$$(K_T^\Delta + V)\alpha = 0$$

where

$$K_T^\Delta(p) = \frac{E_\Delta(p)}{\tanh\left(\frac{E_\Delta(p)}{2T}\right)}, \qquad E_\Delta(p) = \sqrt{(p^2-\mu)^2 + |\Delta(p)|^2}, \qquad \Delta = 2\widehat{V\alpha}.$$

Remark 7.8. For V's with kernel we can of course formulate a gap equation for Δ in much the same way. We get

$$-\Delta(p) = \int_{R^3} \hat V(p,-q)\frac{\Delta(q)}{K_T^\Delta(q)}\,\mathrm{d}q.$$

The normalisation here is different than that of theorem 3.1, since the Fourier transform of the kernel of V is taken in the space $\mathbb{R}^3 \times \mathbb{R}^3$.

Now, to restate theorem 3.7 in this new setting.

Theorem 7.9. *Let $V \in \mathcal{V}$, let $\mu \in \mathbb{R}$ and let $0 \le T < \infty$. Then the following are equivalent*

(i) The normal state Γ_0 is not stable, i.e. $\inf_{\Gamma\in\mathcal{D}} \mathcal{F}(\Gamma) < \mathcal{F}(\Gamma_0)$,

(ii) There exist $\Gamma = (\gamma, \alpha)$ with $\alpha \ne 0$ non-vanishing satisfying the BCS gap equation, $(K_T^\Delta + V)\alpha = 0$

(iii) The linear operator $K_T^0 + V$ has at least one negative eigenvalue.

The proof carries over without change once we prove that

Proposition 7.10. *Let $V \in \mathcal{V}$. Then the essential spectrum of $p^2 + V$ is contained in* $[0, \infty)$.

This follows from (a sufficient generalisation of) [19, Thms. 11.5 and 11.6], which we now state. Define

$$\mathcal{E}(\psi) := \left\langle \psi \middle| p^2 + V \middle| \psi \right\rangle, \qquad E_0 := \inf \left\{ \mathcal{E}(\psi) : \psi \in H^1, \|\psi\|_{L^2} = 1 \right\}$$

Define $\psi_0 \in H^1$ to be an (L^2-normalised) function satisfying $\mathcal{E}(\psi_0) = E_0$, should one such function exist. Define inductively E_k and ψ_k by

$$E_k := \inf \left\{ \mathcal{E}(\psi) : \psi \in H^1, \|\psi\|_{L^2} = 1, \langle \psi | \psi_i \rangle = 0, i = 0, \ldots, k-1 \right\}, \qquad \mathcal{E}(\psi_k) = E_k$$

should they exist.

Proposition 7.11. *Let $V \in \mathcal{V}$.*

- *Suppose that $E_0 < 0$. Then ψ_0 exists, and moreover $(p^2 + V)\psi_0 = E_0 \psi_0$ distributionally.*
- *Suppose that $E_k < 0$ for some k (meaning that also ψ_i exists for all $i < k$). Then ψ_k exists and $(p^2 + V)\psi_k = E_k \psi_k$ distributionally.*

In the sequence of the eigenvalues E_k, each E_k occurs only finitely many times.

It follows that any negative spectral value (i.e. element of the spectrum) of $p^2 + V$ is indeed an eigenvalue with finite multiplicity. Thus, the essential spectrum of $p^2 + V$ is contained in $[0, \infty)$.

The proof is essentially the same as that of [19, Thms. 11.5 and 11.6], only we need to refer to propositions 7.5 and 7.6. instead of [19, Thms. 11.3 and 11.4]. We reproduce a sketch here for convenience.

Proof. First for E_0. Let ψ^j be a minimising sequence of L^2-normalised functions. Then by the bounds in proposition 7.6 we see that $\left\|\psi^j\right\|_{H^1}$ is uniformly bounded, hence for a subsequence it converges weakly to some $\psi_0 \in H^1$. In particular weakly in L^2 and so $\|\psi_0\|_{L^2} \leq 1$. By the weak continuity of V, proposition 7.5, and weak lower semi-continuity of the norm we thus get $\mathcal{E}(\psi_0) \leq \lim \mathcal{E}(\psi^j) = E_0$. Hence

$$E_0 \|\psi_0\|_{L^2} \leq \mathcal{E}(\psi_0) \leq E_0 < 0.$$

So $\|\psi_0\|_{L^2} = 1$ and so ψ_0 is a minimiser. Let now $\phi \in C_0^\infty$ be arbitrary and consider

$$R(\varepsilon) = \frac{\mathcal{E}(\psi_0 + \varepsilon\phi)}{\|\psi_0 + \varepsilon\phi\|_{L^2}^2}.$$

This is a ratio of polynomials in ε and thus differentiable. Its minimum is achieved at $\varepsilon = 0$. Hence, by computing $\frac{\mathrm{d}}{\mathrm{d}\varepsilon} R(\varepsilon)\big|_{\varepsilon=0} = 0$ we get

$$\left\langle \psi_0 \middle| p^2 + V \middle| \phi \right\rangle = E_0 \langle \psi_0 | \phi \rangle .$$

This gives the desired for E_0. The argument for the other energies are similar.

To see that each E_k has finite multiplicity, suppose not, i.e. suppose that there is some k with $E_k = E_{k+1} = \ldots$. Then $\psi_k, \psi_{k+1}, \ldots$ are all mutually orthogonal, hence converge weakly to 0 in L^2. Additionally, their H^1-norms are uniformly bounded, so ψ_j converge weakly to H^1 for a subsequence. This limit must then also be 0. By the weak continuity of V we get

$$E_k = \mathcal{E}(\psi_k) = \lim \mathcal{E}(\psi_j) = \lim \left\langle \psi_j \middle| p^2 \middle| \psi_j \right\rangle + \left\langle \psi_j \middle| V \middle| \psi_j \right\rangle \geq 0.$$

Contradiction. Hence each E_k has only finite multiplicity. □

7.3 The Critical Temperature

One might guess that similar asymptotic formulas as the ones in section 4 hold also in this case. This we have not been able to prove. For the limit of weak coupling the proofs of section 4 are hard to generalise. We sketch here some of the key difficulties.

Firstly, the operator $\mathcal{V}_\mu : L^2(\Omega_\mu) \to L^2(\Omega_\mu)$ is difficult to define. The integral formulation

$$\mathcal{V}_\mu u(p) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{\mu}} \int_{\Omega_\mu} \hat{V}(p-q) u(q) \, \mathrm{d}\omega(q)$$

(here for a potential $V \in L^1 \cap L^{3/2}$) simply does not work. Sure, $\hat{V}$ might be defined (for a $V \in \mathcal{V}$) as a kernel in some L^p-space, but its restriction to a nullset, the Fermi-sphere, is nonsensical. Secondly, the operator $|V|^{1/2}$ does not map into L^1 and so the operator $\mathfrak{F}|V|^{1/2} : L^2 \to L^2(\Omega_\mu)$ is difficult to define.

Additionally, our attempts at generalising lemma 4.5 has failed. That is, decomposing $\frac{1}{K_T}$ as in the proof of lemma 4.5 does not give useful bounds. It is even unclear what reasonable assumptions one could make, such that these proofs would generalise.

For the case of low density there are also many immediate issues. For instance, one needs to find another definition for the scattering length

$$a = \frac{1}{4\pi} \left\langle |V|^{1/2} \middle| \frac{1}{1 + V^{1/2} \frac{1}{p^2} |V|^{1/2}} \middle| V^{1/2} \right\rangle$$

since this definition does not make sense for the more general V's.

8 Ginzburg-Landau Theory

In this section we give a brief recount of the link between BCS theory and Ginzburg-Landau theory. This is presented in [10] and the other papers referenced in [15]. The main takeaway is that in some limit, BCS theory gives Ginzburg-Landau theory. The setting is that the external fields A and W are varying slowly in comparison to the internal interaction V. This is done by introducing a small parameter h and a rescaling. The BCS functional becomes

$$\mathcal{F}(\Gamma) = \operatorname{Tr}\left[\left((-ih\nabla + hA(x))^2 - \mu + h^2 W(x)\right)\gamma\right] - TS(\Gamma) + \iint_{[0,1]^3 \times \mathbb{R}^3} V(h^{-1}(x-y)) |\alpha(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y$$

where the trace is per unit volume and Γ is assumed to be periodic. Additionally we have the assumptions

Assumption 8.1. We assume that W and A are 1-periodic and that their Fourier coefficients satisfy

$$\sum_p |\hat{W}(p)| < \infty, \qquad \sum_p |\hat{A}(p)|(1+|p|) < \infty.$$

In particular W is bounded and A is C^1. Additionally we assume that $V(-x) = V(x)$, that $V \in L^{3/2}$, that $T_c > 0$ and that the operator $K_{T_c} + V$ has a non-zero eigenvector α_0 with eigenvalue 0.

We define the Ginzburg-Landau functional as in [15]

$$\mathcal{E}^{\mathrm{GL}}(\psi) = \int_{[0,1]^3} |(-i\nabla + 2A(x))\psi(x)|^2 + \lambda_1 W(x)|\psi(x)|^2 - \lambda_2 D|\psi(x)|^2 + \lambda_3 |\psi(x)|^4 \, \mathrm{d}x.$$

Here $\psi \in H^1_{\text{per}}$, the periodic functions in H^1_{loc} and $D, \lambda_1 \in \mathbb{R}$ and $\lambda_2, \lambda_3 > 0$. We define a normal state Γ_0 to be a minimiser of the non-interacting BCS functional, i.e. with $V = 0$. We are now ready to state the theorem linking BCS and Ginzburg-Landau theory

Theorem 8.2 ([15, Thm. 4.1], original paper is [10, Thm. 1])**.** *Let $D \in \mathbb{R}$ and $T = T_c(1 - Dh^2)$. Then there exists a $\lambda_0 > 0$ and parameters $\lambda_1, \lambda_2, \lambda_3$ in the Ginzburg-Landau functional such that*

$$\inf_{\Gamma} \mathcal{F}(\Gamma) = \mathcal{F}(\Gamma_0) + \lambda_0 h \inf_{\psi \in H^1_{\text{per}}} \mathcal{E}^{\text{GL}}(\psi) + o(h)$$

as $h \to 0$.

Moreover, if $\Gamma = (\gamma, \alpha)$ satisfies $\mathcal{F}(\Gamma) \le \mathcal{F}(\Gamma_0) + \lambda_0 h \inf_\psi \mathcal{E}^{\text{GL}}(\psi) + o(h)$, i.e. Γ is an approximate minimiser, then there exists a $\psi_0 \in H^1_{\text{per}}$ with $\mathcal{E}^{\text{GL}}(\psi_0) \le \inf_\psi \mathcal{E}^{\text{GL}}(\psi) + o(1)$ such that the Cooper-pair wavefunction α satisfies

$$\|\alpha - \alpha_{\text{GL}}\|^2_{L^2} \le o(1) \|\alpha_{\text{GL}}\|^2_{L^2} = o(1) h^{-1},$$

where α_{GL} is given by

$$\alpha_{\text{GL}}(x, y) = \frac{1}{2h^2} (\psi_0(x) + \psi_0(y)) \frac{1}{(2\pi)^{3/2}} \alpha_0((x - y)/h).$$

9 Conclusion and Perspective

We have here given an overview of the rich mathematical structure relating to the BCS theory of superconductivity. First, in section 2 we setup the model and in section 3 showed both that minimisers exist, and that they satisfy a certain BCS gap equation. We introduced the critical temperature, and showed that it satisfied a certain linear criterion. Next, in section 4 we studied this critical temperature in both the limit of weak coupling and the limit of low density. Here we proved asymptotic formulas for the critical temperature. In section 5 we subsequently studied the energy gap and showed that, both in the limit of weak coupling and in the limit of low density it satisfied an asymptotic formula like that of the critical temperature. Moreover, the ratio of the two converged to the same universal constant in both cases.

After this treatment of the model we considered the validity of the assumptions made. The first to consider was the omission of the direct and exchange terms. In section 6 we included these and saw how the model changed. We considered short-range potentials and showed that for such, this inclusion of the direct and exchange terms in some sense only led to a renormalisation of the chemical potential. Next, in section 7 we dealt with the assumption that the interaction term was given by a multiplication operator. We considered more general operators and showed that the results of section 3 hold in this case also.

Lastly, in section 8 we briefly mentioned the link between BCS theory and Ginzburg-Landau theory. This is also a topic, where much further work could be done. Here we only stated the link for slowly-varying external fields. Studying this model with less restrictive conditions on the external fields is an interesting problem for further study.

Additionally, much work could be done on the validity of the assumptions. In particular, it is interesting in what sense (or even if) the asymptotic results of sections 4 and 5 hold for more general potentials.

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