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# REPRESENTATIONS OF THE POINCARÉ GROUP

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**Abstract**

The purpose of this thesis is to classify the irreducible projective unitary representations of the Poincaré Group. The thesis first gives the needed introductory definitions and results to both understand and solve the problem. The concept of lifting representations is discussed, and it is proven that an equivalent problem is to classify the irreducible unitary representations of the covering group of the Poincaré Group. By the method of induced representations these are classified in the positive mass and positive energy case. Conclusively we provide an example of such induced representation, namely the Dirac Equation for positive mass and spin-1/2.

# Contents

# 1 Introduction

Classifying the irreducible projective unitary representations of the Poincaré Group is a way of combining the special relativistical invariance of the Poincaré Group and the quantum mechanical invariance of unitary and projective unitary operators. The idea of trying to combine special relativity and quantum mechanics is an interesting one, which might cross ones mind after learning of both theories.

The problem is to understand how Poincaré tranformations should transform quantum mechanical systems. An investigation of this naturally leads to the problem of classifying the projective unitary representations of the Poincaré Group. We restrict ourselves to the irreducible projective unitary representations, as these correspond to elementary particles [6, p. 19]. Certain parameters of the representation will then correspond to the mass and spin of the corresponding particle.

Any representation comes with an attached Hilbert space. For the physical interpretation this Hilbert space can be seen as the state-space of the corresponding elementary particle. We will not delve deeper into this interpretation (or any other for that matter) but only quote it as motivation.

We present the subject matter with a lot more detail than the source, [8], on which much of our work is based. We give proofs and more detailed and precise constructions of numerous of the smaller, but nonetheless quite important, and some of the larger results. The proofs of these results in [8] are either missing or lack important details, many of which we provide here. We follow more closely the presentation made in [8] than in [6] as the presentation made in [6] uses results from the general theory of Lie groups and Algebras. The introduction to, and proofs of, these results are beyond the scope of this project.

In the first section (section 2) we introduce the objects of study, in particular the Poincaré group and representation theory. We will also need some Lie group theory, and we conclude the section by giving some elementary results about Lie groups and algebras. we provide proofs of some of the basic results. In particular, we provide a definition for irreducibility in the projective case and discus how irreducibility plays with the lifting of representations. This section is mainly based on [4, 6, 8]. Finally, we provide our own elementary proof of the result that $\mathrm{SO}(3)$ is not simply connected. This is a result we have been unable to find an elementary proof of in the literature.

In the subsequent section (section 3) we investigate how certain properties of the Lie algebra of a Lie group allow us to lift all projective unitary representations of the group to unitary representations of the covering group. We prove that the Lie algebra of the Poincaré Group has this particular property and thus that, in order to classify the irreducible projective unitary representation of the Poincaré Group, we may equivalently classify the irreducible unitary representation of the covering group. We give our own proofs of some of the minor results, in particular how irreducibility works with this lifting and that $\mathrm{SL}_2(\mathbb{C})$ is simply connected. This section is mainly based on [8].

In the penultimate section (section 4) we classify the irreducible unitary representations of the covering group of the Poincaré Group by the method of induced representations. We do not prove the result that any such representation arises (up to equivalence) as an induced one. A proof of this result is beyond the scope of this project, as we have chosen to focus differently. One may prove this result by developing the theory of systems of imprimitivity and linking irreducible unitary representations to systems of imprimitivity, see [1]. Motivated by this result we classify all the induced representation in the positive mass and energy case. This case together with the case of zero mass are the physically most interesting cases. They correspond to ordinary non-zero mass particles and zero-mass particles, for instance photons, respectively [6, p. 63]. Our presentation provides a clearer exposition of this construction and provides a proof for the main lemma for proving the irreducibility. This section is mainly based on [6, 8].

In the final section (section 5) we give an example of an induced representation, namely the Dirac Equation for non-zero mass and spin-1/2. This section is mainly based on [8].

# 2 Introductory Definitions and Results

In this section we give the necessary definitions and introductory results for the background of the main scope of the thesis: "Finding the irreducible representations of the Poincaré group". For this we need to define both what the Poincaré group is and what representations are.

To motivate why this problem is interesting we include the mathematics of quantum mechanics and projective Hilbert spaces.

## 2.1 Special Relativity and the Lorentz Group

We recall some basic notions from special relativity and define the main component of the Poincaré group, namely the Lorentz group. This section is based on [6, p. 1-3] and [8, p. 43-47].

**Definition 2.1.** An element $x = (x^0, \bar{x}) \in \mathbb{R}^4$ is called an event. We say that $x^0$ is the time of the event (we work in units were $c = 1$), and $\bar{x} = (x^1, x^2, x^3)$ is the spatial coordinates of the event.

**Definition 2.2.** The Lorentz metric is the bilinear symmetric function $\langle -, - \rangle : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ defined by

$$\langle x, y \rangle = x^0y^0 - x^1y^1 - x^2y^2 - x^3y^3 = x^0y^0 - \bar{x} \cdot \bar{y}.$$

$\mathbb{R}^4$ equipped with this "metric" is called spacetime or Minkowski space.

**Definition 2.3.** An event $x \in \mathbb{R}^4$ is

- time-like if $\langle x, x \rangle > 0$,
- light-like if $\langle x, x \rangle = 0$,
- space-like if $\langle x, x \rangle < 0$.

Note that we may express $\langle -, - \rangle$ in terms of the Minkowski metric $\eta = (\eta_{\mu\nu})$,

$$\eta_{\mu\nu} = \begin{cases} 1, & \text{if } \mu = \nu = 0 \\ -1, & \text{if } \mu = \nu \neq 0 \\ 0, & \text{if } \mu \neq \nu. \end{cases}$$

Then $\langle x, y \rangle = \sum_{\mu,\nu} g_{\mu\nu} x^\mu y^\nu$. Thinking of $\eta$ as a matrix, this is $\langle x, y \rangle = x^T \eta y$.

**Proposition 2.4.** The relation $\sim$ on the time-like events $\{x \in \mathbb{R}^4 | \langle x, x \rangle > 0\}$ defined by $x \sim y$ if $\langle x, y \rangle > 0$ is an equivalence relation with two equivalence classes.

*Proof.* The relation is clearly reflexive and symmetric. For transitivity note the following. For any $x, y$ time-like events, $x^0, y^0 \neq 0$. Moreover, the defining property gives that $(x^0)^2 - \|\bar{x}\|^2 > 0$. Suppose that $\operatorname{sgn} x^0 = \operatorname{sgn} y^0$. Then $x^0y^0 > 0$ and we get by Cauchy Schwartz (on the regular $\mathbb{R}^3$ inner product)

$$(\bar{x} \cdot \bar{y})^2 \leq \|\bar{x}\|^2 \, \|\bar{y}\|^2 < (x^0)^2 (y^0)^2.$$

By extracting square roots we conclude that $\langle x, y \rangle > 0$. Similarly we may get that if $\operatorname{sgn} x^0 \neq \operatorname{sgn} y^0$ then $\langle x, y \rangle < 0$. Hence all $x$ with the same sign zeroth coordinate are related and those with different sign zeroth coordinate are not. It follows that $\sim$ is transitive. The equivalence classes are those with positive resp. negative zeroth coordinate. $[(1, 0, 0, 0)]$ resp. $[(-1, 0, 0, 0)]$. Since any timelike $x$ has $x^0 \neq 0$ we see that these are all the equivalence classes. □

The equivalence classes in the above proposition, $[(1, 0, 0, 0)]$ and $[(-1, 0, 0, 0)]$, are called the future resp. the past corresponding to how a particle at rest will have worldline $(t, 0, 0, 0)$, $t \in \mathbb{R}$, with positive zeroth coordinate for future times $t$.

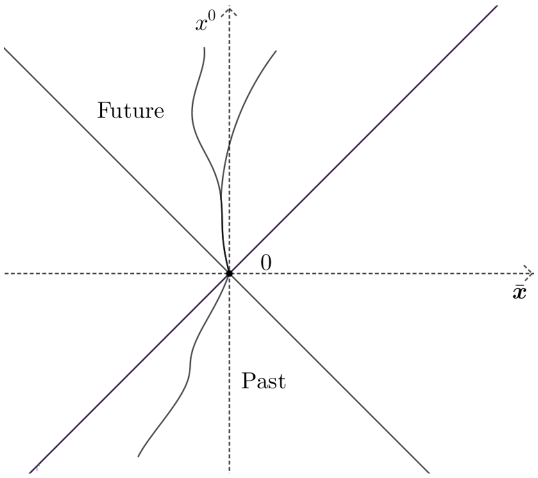

Figure 1: The future and the past. Drawn are also different worldlines for a particle. The causality is upwards. The light-cones (for the point $(0,0,0,0)$) are drawn as well. They are the boundaries for the time-like events.

**Definition 2.5.** A Lorentz transformation is a linear map $\Lambda : \mathbb{R}^4 \to \mathbb{R}^4$ preserving the "distance" i.e.

$$\langle \Lambda x, \Lambda y \rangle = \langle x, y \rangle$$

for all $x, y \in \mathbb{R}^4$. With the metric $\eta$ this is equivalent to the equations (by letting $x, y$ run through a basis)

$$\sum_{\mu,\nu} \Lambda^{\mu}{}_{\rho} \eta_{\mu\nu} \Lambda^{\nu}{}_{\tau} = \eta_{\rho\tau} \tag{1}$$

for $\rho, \tau \in \{0, 1, 2, 3\}$ i.e. $\Lambda^T \eta \Lambda = \eta$ if we think of $\eta$ as a matrix with the same indices as $\eta$. These equations are all written out in full in the appendix, see section A.1.

When we assume that light has a finite speed (here $c = 1$), which is the same in all (inertial) coordinate systems, we get the results from special relativity, stating that any coordinate transformation must preserve the Lorentz metric. The classical transformations, the Galilei transformations [9, p. 1226], does not preserve this metric hence we need to consider another type of coordinate transformations, namely the Lorentz transformations.

The equations (1) are in particular for $\rho = \tau = 0$

$$(\Lambda^0{}_0)^2 - (\Lambda^1{}_0)^2 - (\Lambda^2{}_0)^2 - (\Lambda^3{}_0)^2 = 1.$$

Since $\Lambda$ has real entries we conclude that $\Lambda^0{}_0 \geq 1$ or $\Lambda^0{}_0 \leq -1$.

Thinking of $\eta$ as a matrix we get

$$1 = -\det \eta = -\det(\Lambda^T \eta \Lambda) = -\det \eta (\det \Lambda)^2 = (\det \Lambda)^2.$$

We conclude that $\Lambda$ is invertible and moreover that $\det \Lambda = \pm 1$.

Note that the Lorentz transformations with the composition of matrix multiplication form a group. This is a straightforward calculation

**Definition 2.6.** The group of Lorentz transformations is called the Lorentz group and is denoted $\mathcal{L}$.

Note that $\mathcal{L}$ is not connected: Define $f : \mathcal{L} \to \mathbb{R}$ by $f(\Lambda) = \Lambda^0{}_0$. Then $f$ is continuous so both $f^{-1}((-\infty, 0))$ and $f^{-1}((0, \infty))$ are open and since $\Lambda^0{}_0 \neq 0$ we have that $\mathcal{L} = f^{-1}((-\infty, 0)) \cup f^{-1}((0, \infty))$. Both of these are non-empty as $I_4, -I_4 \in \mathcal{L}$ are elements of the first resp. the second set.

Note that by definition the Lorentz group leaves timelike events as timelike and similarly for space- and lightlike.

We will restrict ourselves to the Lorentz transformations preserving the direction of time. Working with the possibility of reversing time is not a major complication, but we will nonetheless refrain from considering such transformations. It is beyond the scope of this project to also consider time reversal. We will thus restrict ourselves to the groups $\mathcal{L}^{\uparrow}$, the orthochronous Lorentz group, and $\mathcal{L}_+^{\uparrow}$, the restricted Lorentz group [6, p. 2]. The group of Lorentz transformations preserving that equivalence classes of future and past (i.e. it is not time-reversing) resp. also of determinant 1.

$$\mathcal{L}^{\uparrow} = \{\Lambda \in \mathcal{L} | {\Lambda^0}_0 \geq 1\}, \qquad \mathcal{L}_+^{\uparrow} = \{\Lambda \in \mathcal{L} | \det \Lambda = 1, {\Lambda^0}_0 \geq 1\}.$$

Most of what we do is about $\mathcal{L}_+^{\uparrow}$ and the extension to $\mathcal{L}^{\uparrow}$ is easy. An extension to $\mathcal{L}$ is similarly not so complicated, see [8, p. 75].

**Proposition 2.7.** $\mathcal{L}_+^{\uparrow}$ is the (path-)connected component of the identity in $\mathcal{L}$.

In order to prove this we first give a full classification of the Lorentz group.

Note first that we may identify $SO(3)$ as a subgroup of $\mathcal{L}_+^{\uparrow}$ by the injective group homomorphism

$$\Lambda^R : SO(3) \hookrightarrow \mathcal{L}$$
$$R \mapsto \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}$$

Describing the rotation that is $R$ by a rotation vector $\bar{\varphi}$ of length $\leq \pi$ and direction the axis of rotation, we get Lorentz transformations $\Lambda^{\varphi}(\bar{\varphi})$. The reason for this parametrization is that we get the equation $\Lambda^{\varphi}(\bar{\varphi} + \bar{\varphi}') = \Lambda^{\varphi}(\bar{\varphi})\Lambda^{\varphi}(\bar{\varphi}')$ if $\bar{\varphi} \| \bar{\varphi}'$ are parallel.

Recall that for a particle with velocity $\bar{v}$ we define the $\gamma$-factor $\gamma(v) = \frac{1}{\sqrt{1-v^2}}$.

**Definition 2.8.** The Lorentz boost with velocity $\bar{v}$ is

$$\Lambda^v(\bar{v}) = \begin{bmatrix} \gamma(v) & \gamma(v)\bar{v}^T \\ \gamma(v)\bar{v} & I_3 + \frac{\gamma(v)-1}{v^2}\bar{v}\bar{v}^T \end{bmatrix}.$$

One can check that $\Lambda^v(\bar{v}) \in \mathcal{L}_+^{\uparrow}$. Parameterizing the velocities with the hyperbolic coordinates $\bar{\omega} = \operatorname{arctanh}(v)\hat{v}$ we get Lorentz transformations $\Lambda^{\omega}(\bar{\omega})$, $\gamma(v) = \cosh \omega$. Again, the reason for this parametrization is that we have linearity, $\Lambda^{\omega}(\bar{\omega} + \bar{\omega}') = \Lambda^{\omega}(\bar{\omega})\Lambda^{v}(\bar{\omega}')$ for parallel $\bar{\omega} \| \bar{\omega}'$.

**Proposition 2.9.** For any $\Lambda \in \mathcal{L}_+^{\uparrow}$ there exists unique $\bar{v} \in \mathbb{R}^3, R \in SO(3)$ such that $\Lambda = \Lambda^v(\bar{v})\Lambda^R(R)$. The assignment $\Lambda \mapsto (\bar{v}, R)$ is continuous.

*Proof.* We prove this result in the appendix, see section A.2. □

This proves that $\mathcal{L}_+^{\uparrow}$ is in fact path-connected. Given $\Lambda \in \mathcal{L}_+^{\uparrow}$ write $\Lambda = \Lambda^{\varphi}(\bar{\varphi})\Lambda^{\omega}(\bar{\omega})$. Then the path $t \mapsto \Lambda^{\varphi}(t\bar{\varphi})\Lambda^{\omega}(t\bar{\omega})$ connects $I_4$ and $\Lambda$ in $\mathcal{L}_+^{\uparrow}$.

Any other Lorentz transformation has either ${\Lambda^0}_0 \leq -1$ or $\det \Lambda = -1$. In both cases they are in a different component than $I_4$.

However, $\mathcal{L}_+^{\uparrow}$ is not simply connected since we have the following

**Proposition 2.10.** $SO(3)$ is not simply connected.

*Proof.* We have been unable to find an elementary proof of this result in the literature. Hence we provide our own elementary proof of this result in section 2.6. □

The intuition one should have about this is that since rotations by $\pi$ and $-\pi$ are the same, a curve in $SO(3)$ may use this identification to make a "jump". Such jumps can't be removed continuously. The proof is long and technical, which is why we postpone it to the end of the section. It is not needed to understand the rest of this project.

Using this we get the following

**Proposition 2.11.** $\mathcal{L}_+^{\uparrow}$ is not simply connected.

*Proof.* Suppose for contradiction that $\mathcal{L}_+^{\uparrow}$ is simply connected. Define by Prop. 2.10 a closed path $\gamma(t) \in SO(3) \leq \mathcal{L}_+^{\uparrow}$ s.t. $\gamma$ is not contractible in $SO(3)$. By assumption $\gamma(t)$ is contractible in $\mathcal{L}_+^{\uparrow}$. Let $\Lambda(s,t)$ be a homopoty contracting $\gamma$ to a point. ($\Lambda(0,t) = \gamma(t)$). Now by Prop. 2.9 we may write $\Lambda(s,t) = \Lambda^v(\bar{v}(s,t))\Lambda^R(R(s,t))$ with $R(s,t)$ and $\bar{v}(s,t)$ continuous. Hence $R(s,t)$ is a homotopy in $SO(3)$ contracting $\gamma$ to a point. Contradiction. We conclude that $\mathcal{L}_+^{\uparrow}$ is not simple connected. □

**Definition 2.12.** The discrete Lorentz transformations

$$P = \begin{bmatrix} 1 & 0 \\ 0 & -I_3 \end{bmatrix}, \qquad T = \begin{bmatrix} -1 & 0 \\ 0 & I_3 \end{bmatrix}, \qquad PT = TP = -I_4, \qquad I_4$$

together form a group called the discrete Lorentz group. $P$ and $T$ are called resp. space inversion and time reversal.

With these we get that $\mathcal{L}^{\uparrow} = \{\Lambda, P\Lambda | \Lambda \in \mathcal{L}_+^{\uparrow}\}$ and $\mathcal{L} = \{\Lambda, T\Lambda | \Lambda \in \mathcal{L}^{\uparrow}\}$. Hence with little work we may extend results for $\mathcal{L}_+^{\uparrow}$ to $\mathcal{L}^{\uparrow}$ and (if we wanted to) $\mathcal{L}$.

## 2.2 The Poincaré Group

We are now ready to define the group of interest.

**Definition 2.13.** A Poincaré transformation $\Pi = (a, \Lambda)$ with $a \in \mathbb{R}^4, \Lambda \in \mathcal{L}$ is a map $\Pi : \mathbb{R}^4 \to \mathbb{R}^4$ defined by $\Pi(x) = \Lambda x + a$.

The set $\mathcal{P} = \{\Pi = (a, \Lambda) | a \in \mathbb{R}^4, \Lambda \in \mathcal{L}\}$ is a group with composition

$$(a, \Lambda) \circ (a', \Lambda') = (a + \Lambda a', \Lambda\Lambda').$$

That is $\mathcal{P} = \mathbb{R}^4 \rtimes \mathcal{L}$ is the semi-direct product. The group $\mathcal{P}$ is called the Poincaré group. (This composition comes from the composition of the maps $\Pi, \Pi' : \mathbb{R}^4 \to \mathbb{R}^4$.) Note that inversion is given by

$$(a, \Lambda)^{-1} = (-\Lambda^{-1} a, \Lambda^{-1}).$$

Similarly as for $\mathcal{L}_+^{\uparrow}$ we define the orthochronous resp. restricted Poincaré group

$$\mathcal{P}^{\uparrow} = \mathbb{R}^4 \rtimes \mathcal{L}^{\uparrow}, \qquad \mathcal{P}_+^{\uparrow} = \mathbb{R}^4 \rtimes \mathcal{L}_+^{\uparrow}.$$

As before $\mathcal{P}_+^{\uparrow}$ is the connected component of the identity $(0, I_4)$ in $\mathcal{P}$.

## 2.3 Quantum Mechanics and Projective Spaces

We define the relevant projective space of Quantum Mechanics. This section is based on [8, p. 50-51]

In Quantum Mechanics one of the postulates is that the state of a physical system is an element of norm 1 in some Hilbert space $(\mathfrak{H}, \langle -, - \rangle)$, see [2, p. 1].

Now the states $\psi, \phi \in \mathfrak{H}$ give rise to the same probabilities of observables if they are scalar multiples of each other, $\phi = \lambda\psi$ for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Since we are not working with constructing superpositions of states we will identify such states and work with the equivalence classes of this identification instead. We are thus lead to the following.

**Definition 2.14.** Define the equivalence relation $\sim$ on $\mathfrak{H}$ by $\psi \sim \phi$ if $\phi = \lambda\psi$ for some $\lambda \in \mathbb{C} \backslash \{0\}$. The quotient space $(\mathfrak{H} \backslash \{0\})/\sim$ is called the projective space and is denoted by $\widehat{\mathfrak{H}}$.

We equip $\mathfrak{H}$ with its norm topology and $\widehat{\mathfrak{H}}$ with the quotient topology. Note that we do not require $|\lambda| = 1$ in the identification. This can be seen as just normalizing all the elements prior to the identification above with $|\lambda| = 1$.

**Definition 2.15.** On $\widehat{\mathfrak{H}}$ we define the bilinear map $\langle -, - \rangle : \widehat{\mathfrak{H}} \times \widehat{\mathfrak{H}} \to \mathbb{R}$ by

$$\left\langle \widehat{\phi}, \widehat{\psi} \right\rangle = \frac{|\langle \phi, \psi \rangle_{\mathfrak{H}}|^2}{\|\phi\|_{\mathfrak{H}}^2 \, \|\psi\|_{\mathfrak{H}}^2}.$$

It is the transition probability of passing from state $\phi$ to state $\psi$.

Note that the map defined above is well-defined, since we scale by the norm of the representatives $\phi, \psi$.

We are only interested in transformations preserving these transition probabilities. The transformations should preserve the laws of (quantum) physics and hence in particular the transition probabilities.

**Definition 2.16.** A bijection $T : \widehat{\mathfrak{H}} \to \widehat{\mathfrak{H}}$ is called a symmetry transformation or an automorphism if it preserves all transition probabilities i.e.

$$\left\langle T\widehat{\phi}, T\widehat{\psi} \right\rangle = \left\langle \widehat{\phi}, \widehat{\psi} \right\rangle$$

for all $\widehat{\phi}, \widehat{\psi} \in \widehat{\mathfrak{H}}$.

Through ordinary function composition all such transformations form a group $\mathrm{Aut}(\widehat{\mathfrak{H}})$ called the automorphism group.

Suppose that $A : \mathfrak{H} \to \mathfrak{H}$ is (anti)unitary. Then we define $\widehat{A} : \widehat{\mathfrak{H}} \to \widehat{\mathfrak{H}}$ by $\widehat{A}\widehat{\psi} = \widehat{A\psi}$. Note that the map $\widehat{A}$ is well-defined: If $A$ is unitary we have $\widehat{A\lambda\psi} = \widehat{\lambda A\psi} = \widehat{A\psi}$ and similarly if $A$ is antiunitary.

Moreover we have that

$$\left\langle \widehat{A}\widehat{\phi}, \widehat{A}\widehat{\psi} \right\rangle = \frac{|\left\langle A\phi, A\psi \right\rangle_{\mathfrak{H}}|^2}{\|A\phi\|_{\mathfrak{H}}^2 \, \|A\psi\|_{\mathfrak{H}}^2} = \frac{|\left\langle \phi, \psi \right\rangle_{\mathfrak{H}}|^2}{\|\phi\|_{\mathfrak{H}}^2 \, \|\psi\|_{\mathfrak{H}}^2} = \left\langle \widehat{\phi}, \widehat{\psi} \right\rangle.$$

So $\widehat{A}$ is a symmetry transformation.

**Definition 2.17.** We define the groups

$$\begin{aligned} \tilde{U}(\mathfrak{H}) &= \{A \in GL(\mathfrak{H}) | A \text{ unitary or antiunitary}\} \\ U(\mathfrak{H}) &= \{A \in GL(\mathfrak{H}) | A \text{ unitary}\} \leq \tilde{U}(\mathfrak{H}). \end{aligned}$$

We equip these groups with the strong operator topology, i.e. the weakest topology making all evaluations

$$e_x : \tilde{U}(\mathfrak{H}) \ni A \mapsto Ax \in \mathfrak{H}, \qquad x \in \mathfrak{H}$$

continuous.

Note that $U(\mathfrak{H}) \leq \tilde{U}(\mathfrak{H})$ is a subgroup of index 2, since a product of any two antiunitary operators is unitary.

With this we will by $p : \tilde{U}(\mathfrak{H}) \to \mathrm{Aut}(\widehat{\mathfrak{H}})$ denote the above map, $A \mapsto \widehat{A}$. With this notation we have the following

**Theorem 2.18** (Wigner, [8, Thm 2.7])**.** *The map* $p$ *defined above is surjective and has kernel* $\ker p = \{e^{i\theta} \, \mathrm{id}_{\mathfrak{H}} \,|\, 0 \leq \theta < 2\pi\}$.

## 2.4 Representation Theory

We want to combine the relativistic invariance of $\mathcal{P}_+^\uparrow$ and $\mathcal{P}^\uparrow$ with the quantum mechanical invariance of $\mathrm{Aut}(\widehat{\mathfrak{H}})$. This should correspond to $\mathcal{P}_+^\uparrow$ and $\mathcal{P}^\uparrow$ acting on $\widehat{\mathfrak{H}}$, specifically a map $\rho : \mathcal{P}_+^\uparrow \to \mathrm{Aut}(\widehat{\mathfrak{H}})$ with certain properties. These will be described below. This section is based on [8, p. 51-54].

**Definition 2.19.** Let $U(\widehat{\mathfrak{H}}) = p(U(\mathfrak{H}))$ and equip it with the quotient topology from the surjection $p$ i.e. the topology making all evaluations continuous.

**Definition 2.20.** Let $\mathcal{G}$ be a topological group and $\mathfrak{H}$ a Hilbert space.

- A projective representation $\rho$ of $\mathcal{G}$ in $\mathfrak{H}$ is a continuous group homomorphism $\rho : \mathcal{G} \to \mathrm{Aut}(\widehat{\mathfrak{H}})$, in particular an action of $\mathcal{G}$ on $\widehat{\mathfrak{H}}$.

  If moreover $\rho(\mathcal{G}) \subset U(\widehat{\mathfrak{H}})$ we say that $\rho$ is a projective unitary representation
- A representation $\pi$ of $\mathcal{G}$ in $\mathfrak{H}$ is a continuous group homomorphism $\pi : \mathcal{G} \to GL(\mathfrak{H})$, i.e. an action of $\mathcal{G}$ on $\mathfrak{H}$.

  If moreover $\pi(\mathcal{G}) \subset U(\mathfrak{H})$ we say that $\pi$ is a unitary representation.

**Definition 2.21.** Two representations $\pi, \pi'$ of $\mathcal{G}$ in the Hilbert spaces $\mathfrak{H}, \mathfrak{H}'$ are said to be equivalent if there exists some unitary map $U : \mathfrak{H} \to \mathfrak{H}'$ s.t. $\pi'(g) = U\pi(g)U^*$ for all $g \in \mathcal{G}$.

Given a (unitary) representation $\pi$ we may construct a projective (unitary) representation $\rho = p \circ \pi$. Conversely

**Definition 2.22.** Let $\rho$ be a projective (unitary) representation. We say that $\rho$ admits a lifting $\pi$ if $\rho = p \circ \pi$ for some (unitary) representation $\pi$.

This is not a weak property. The following example shows, that this does not hold for all projective representations.

**Example 2.23** ([8, p. 53])**.** Consider the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong \{I_4, P, T, PT\}$, the Discrete Lorentz Group, and the projective representation in $\mathbb{C}^2$ defined by

$$\rho(P) = \widehat{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}, \quad \rho(T) = \widehat{\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}}, \quad \rho(PT) = \widehat{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$$

Then this representation does not admit a lifting.

*Proof.* Write

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

These are the Pauli matrices. It is straightforward to verify the equalities

$$\sigma_j \sigma_k = \delta_{jk} + i \sum_m \varepsilon_{jkm} \sigma_m$$

where $\varepsilon_{jkm}$ is the Levi-Civita symbol. Suppose that $\rho = p \circ \pi$ for some representation $\pi$. Then by Wigner's Thm. (Thm. 2.18) we have that

$$\pi(P) = \lambda_1 \sigma_1, \quad \pi(T) = \lambda_2 \sigma_2, \quad \pi(PT) = \lambda_3 \sigma_3.$$

for some $\lambda_1, \lambda_2, \lambda_3$ all with norm 1.

Then since $PT = TP$ we have that

$$i\lambda_1\lambda_2\sigma_3 = \pi(P)\pi(T) = \pi(T)\pi(P) = -i\lambda_1\lambda_2\sigma_3.$$

But $\sigma_3 \neq 0$. Contradiction. Hence $\rho$ does not admit a lifting. ☐

Since $SO(3)$ contains a copy of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, namely the rotations around the coordinates axes with angle $\pi$, this proves that if we may extend a projective representation of this subgroup to the whole of $SO(3)$, then $SO(3)$ also has a projective representation, which does not admit a lifting.

**Definition 2.24.** Let $\rho$ be a projective representation. We say that a nonzero subspace $\{0\} \neq V = \mathrm{span}_{\mathbb{C}}\{\psi_i \,|\, i \in I\} \subset \mathfrak{H}$ is invariant for $\rho$ if, for all $g \in \mathcal{G}$, we have that $\mathrm{span}_{\mathbb{C}}\{\phi_i | i \in I\} \subset \mathrm{span}_{\mathbb{C}}\{\psi_i | i \in I\}$, where $\widehat{\phi_i} = \rho(g)(\widehat{\psi_i})$. (Note that this does not depend on the choice of representatives $\phi_i$.).

**Definition 2.25.** We say that the representation $\pi$ is

- reducible if there exist some non-zero proper subspace $\{0\} \neq V \subsetneq \mathfrak{H}$ such that $\pi(g)(V) \subset V$ for all $g \in \mathcal{G}$.
- irreducible if it is not reducible.

We say that the projective representation $\rho$ is

- reducible if there exists a proper invariant subspace for $\rho$.
- irreducible if it is not reducible.

One might wonder how the irreducibility plays with liftings. We have the following

**Proposition 2.26.** Suppose that $\rho = p \circ \pi$ where $\rho$ is a projective representation and $\pi$ is a representation. Then $\rho$ is irreducible if and only if $\pi$ is.

*Proof.* Let $\{0\} \neq V = \mathrm{span}_{\mathbb{C}}\{\psi_i\}$ be an invariant subspace for $\rho$. Then for any $g \in \mathcal{G}$ we may pick the representatives $\phi_i = \pi(g)\psi_i$ since we have

$$\widehat{\pi(g)\psi_i} = (p(\pi(g)))\widehat{\psi_i} = \rho(g)(\widehat{\psi_i}).$$

Now irreducibility of $\pi$ and $\rho$ are both equivalent to the implication

$$\mathrm{span}_{\mathbb{C}}\{\phi_i\} \subset V \Rightarrow V = \mathfrak{H}.$$ ☐

We are interested in the irreducible projective unitary representations of $\mathcal{L}^{\uparrow}$ and $\mathcal{L}_{+}^{\uparrow}$ in $\mathfrak{H}$. The projective part is exactly that we get symmetry transformations, which we wanted for the transformations to preserve quantum physics. In physics it is the unitary operators, which are of most importance, which is why we restrict ourselves to unitary operators.

In some sense a pair $(\rho, \mathfrak{H})$ of an irreducible unitary projective representation and the corresponding Hilbert space is an elementary particle [6, p. 19]. This is why we will only consider the irreducible representations. The space $\mathfrak{H}$ can be seen as the state-space of the corresponding elementary particle.

The difference between working with $\mathcal{L}_{+}^{\uparrow}$ and $\mathcal{L}^{\uparrow}$ is minimal, and one might as well consider time inversions and work with $\mathcal{L}$.

From general representation theory we have the following

**Theorem 2.27** (Schur's Lemma, [8, Lem. 2.12])**.** *If $\pi$ is a unitary representation then the following are equivalent*

- *$\pi$ is irreducible*
- *If $T : \mathfrak{H} \to \mathfrak{H}$ is bounded linear satisfies $T\pi(g) = \pi(g)T$ for all $g \in \mathcal{G}$ then there exists $\lambda \in \mathbb{C}$ s.t. $T = \lambda\, \mathrm{id}_{\mathfrak{H}}$.*

## 2.5 Lie Groups and Algebras

The final introductory definitions we need are that of Lie groups and algebras, which will prove useful when discussing the possibility of lifting projective (unitary) representations. This section is based on [4].

In order to give the definition of a Lie group in full generality, one needs to know about smooth manifolds from differential geometry. To also give a full introduction to this subject is beyond the scope of this project. We will only state some of the main results for matrix groups (i.e. $\mathcal{G} \leq \mathrm{GL}_n(\mathbb{K})$ for $\mathbb{K} = \mathbb{R}$ of $\mathbb{K} = \mathbb{C}$).

**Definition 2.28.** A Lie group is a group $\mathcal{G}$ equipped with the structure of a smooth manifold s.t. multiplication and inversion are both smooth maps.

For our purposes it is enough that matrix groups, $\mathbb{R}^4$ and semidirect products of such are Lie groups. Since the Lorentz group is a matrix group, we see that it is a Lie group.

For matrices we have the exponential map

$$\exp : A \mapsto \exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

This sum is readily checked to be absolutely convergent. Moreover, if $AB = BA$, it is a straightforward computation to show that $\exp(A + B) = \exp(A)\exp(B)$. In addition, the function

$$t \mapsto \exp(tA) = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$$

is seen to be differentiable with derivative $\exp(tA)A$ as the above sum is uniformly convergent (in $t$) on any compact set.

We define the operator norm by $\|A\| = \sup\{\|Ax\| : \|x\| \leq 1\}$ and the topology on matrix groups by this norm.

**Definition 2.29.** A one-parameter subgroup of a matrix Lie group $\mathcal{G}$ is a continuous group homomorphism $\alpha : (\mathbb{R}, +) \to \mathcal{G}$.

It is clear that $t \mapsto \exp(t\mathbf{A})$ for some fixed matrix $\mathbf{A}$ is a one-parameter subgroup. In fact we have the contrary as well

**Proposition 2.30** ([4, Thm. 8])**.** Suppose $\alpha$ is a one-parameter subgroup of a matrix Lie group $\mathcal{G}$. Then there exists a unique matrix $\mathbf{A}$ such that $\alpha(t) = \exp(t\mathbf{A})$ for all $t$.

**Definition 2.31.** Let $\alpha(t) = \exp(t\mathbf{A})$ be a one-parameter subgroup. Then $\mathbf{A} = \frac{d}{dt}\alpha(t)\big|_{t=0}$ is said to be the infinitesimal generator of $\alpha$.

**Definition 2.32.** A Lie algebra is a vector space $\mathfrak{g}$ over a field (which in our case always will be $\mathbb{R}$) equipped a bilinear map $[-,-]$ called the Lie bracket satisfying

$$\begin{aligned}&[\mathbf{A},\mathbf{B}] = -[\mathbf{B},\mathbf{A}]\\&[[\mathbf{A},\mathbf{B}],\mathbf{C}] + [[\mathbf{B},\mathbf{C}],\mathbf{A}] + [[\mathbf{C},\mathbf{A}],\mathbf{B}] = 0 \qquad \text{(Jacobian identity).}\end{aligned}$$

**Definition 2.33.** A Lie algebra homomorphism between to Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ is a linear map $\varphi : \mathfrak{g} \to \mathfrak{g}'$ s.t. $\varphi([\mathbf{A},\mathbf{B}]) = [\varphi(\mathbf{A}), \varphi(\mathbf{B})]$ for all $\mathbf{A},\mathbf{B} \in \mathfrak{g}$.

The main result is that to a Lie group we may assign a Lie algebra. This we will only state.

**Theorem 2.34.** *Let $\mathcal{G} \leq \mathrm{GL}_n(\mathbb{K})$ be a matrix group. Here $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Then*

$$L_{\mathcal{G}} = \{\mathbf{A} \in M_{n\times n}(\mathbb{K}) \mid \exp(t\mathbf{A}) \in \mathcal{G} \text{ for all } t \in \mathbb{R}\}$$

*with $[\mathbf{A},\mathbf{B}] := \mathbf{AB} - \mathbf{BA}$ is a Lie algebra called the Lie algebra of $\mathcal{G}$.*

**Theorem 2.35** (Baker-Campbell-Hausdorff [4, p. 4])**.** *For $\|\mathbf{A}\|, \|\mathbf{B}\|$ sufficiently small there exists a solution $\mathbf{A}*\mathbf{B}$ to the equation $\exp(\mathbf{X}) = \exp(\mathbf{A})\exp(\mathbf{B})$. $\mathbf{A}*\mathbf{B}$ is unique if we assume that $\|\mathbf{X}\|$ is small. We have the following formula*

$$\mathbf{A} * \mathbf{B} = \mathbf{A} + \mathbf{B} + \frac{1}{2}[\mathbf{A},\mathbf{B}] + \dots$$

*where the higher order terms are obtained by iterative applications of the Lie bracket.*

**Theorem 2.36.** *Suppose that $\mathcal{G}$ is a matrix Lie group and $q_1, \dots, q_s$ are one-parameter subgroups with infinitesimal generators $\mathbf{A}_1, \dots, \mathbf{A}_s$ such that any $g \in \mathcal{G}$ can be written $g = q_1(t_1)\cdots q_s(t_s)$ for some $t_1, \dots, t_s \in \mathbb{R}$. Suppose moreover that for all $j,k$ we have that $[\mathbf{A}_j, \mathbf{A}_k] \in \mathrm{span}_{\mathbb{R}}\{\mathbf{A}_1, \dots, \mathbf{A}_s\}$.*

*Then $L_{\mathcal{G}} = \mathrm{span}_{\mathbb{R}}\{\mathbf{A}_1, \dots, \mathbf{A}_s\}$.*

*Proof.* We give a sketch in the case $s = 2$.

Let $\mathbf{B} \in L_{\mathcal{G}}$ then $\exp(t\mathbf{B}) \in \mathcal{G}$ for all $t \in \mathbb{R}$ so by assumption we may write that $\exp(t\mathbf{B}) = \exp(t_1\mathbf{A}_1)\exp(t_2\mathbf{A}_2)$ for some $t_1, t_2 \in \mathbb{R}$ (for each $t$). Let now $t$ be sufficiently small. By Baker-Campbell-Hausdorff then $\mathbf{B} = \frac{1}{t}(t_1\mathbf{A}_1)*(t_2\mathbf{A}_2) \in \overline{\mathrm{span}\{\mathbf{A}_1, \mathbf{A}_2\}}$. (There might be some convergence issues here if the coefficients of $[\mathbf{A}_1, \mathbf{A}_2]$ written as a linear combination are large. They are all ±1 in the cases we will consider.) Since $s = 2$ is finite the closure is superfluous and we get the desired. ☐

## 2.6 SO(3) is not simply connected

We give our own elementary proof of Prop. 2.10.

The intuition one should have is that a curve may "jump", i.e. use that rotation by an angle of $\pi$ is the same as rotation by an angle of $-\pi$. The intuition again is that such jumps can't be removed continuously. Hence that a closed curve with a jump can't be contracted to a point.

The difficulty in proving this result is to find a precise way of defining what a jump is, in such a way, that we may use only the continuity to prove that jumps can't disappear. This is the reason for the proof to be so long and technical.

The idea for this first part was given to me by my advisor Jan Philip Solovej.

### The homeomorphism $SO(3) \cong \overline{B_{\mathbb{R}^3}}(0,1)/\sim$

Here the relation $\sim$ is given by $x \sim -x$ if $\|x\| = 1$. This extends to an equivalence relation by declaring $y \sim y$ for all $y \in \overline{B_{\mathbb{R}^3}}(0,1)$.

We first describe the map $SO(3) \to \overline{B_{\mathbb{R}^3}}(0,1)/\sim$.

Let $R \in SO(3)$. Consider the characteristic polynomial $p(\lambda) = \det(R - \lambda I_3)$. $p$ is of degree 3 so it has at least 1 root over $\mathbb{R}$, $\lambda_1$. Over $\mathbb{C}$ we have the roots $\lambda_1, \lambda_2, \lambda_3$. Now, $p$ is defined over $\mathbb{R}$ so either $\lambda_2, \lambda_3 \in \mathbb{R}$ or $\lambda_3 = \overline{\lambda_2}$ is the complex conjugate.

Since $R$ is an isometry we have that $|\lambda_1| = 1$ i.e. $\lambda_1 = \pm 1$. And the same proves that if $\lambda_2, \lambda_3$ are real, then $\lambda_2, \lambda_3 = \pm 1$.

Since $R^t = R^{-1}$ we see that R is normal and hence diagonizable over $\mathbb{C}$. This proves that $1 = \det R = \lambda_1\lambda_2\lambda_3$. In the case $\lambda_2, \lambda_3 \in \mathbb{R}$ we conclude that at least 1 eigenvalue is 1, since they can't all be $-1$. If $\lambda_2, \lambda_3 \notin \mathbb{R}$ we have $\lambda_2\lambda_3 = |\lambda_2|^2 > 0$ so $\lambda_1 = \frac{1}{|\lambda_2|^2} > 0$. We conclude that $\lambda_1 = 1$ and so we have an eigenvalue of 1 in this case also.

Define $\widehat{n}$ as the corresponding eigenvector. Note that $R \mapsto \widehat{n} \in \mathbb{S}^2$ is continuous as it involves solving a linear system of equations $(R - I_3)\widehat{n} = 0$.

writing R in an orthonormal basis with $\widehat{n}$ as the first vector (note that this is a continuous transformation of R) we see that

$$R = \begin{pmatrix} 1 & 0 \\ 0 & R_2 \end{pmatrix}$$

where $R_2 \in SO(2)$. Note that $R \mapsto (\widehat{n}, R_2)$ is a bijection (with some rule for choosing $\widehat{n}$ and the order of the basis) onto $\mathbb{S}^2 \times SO(2)$. Now $R_2(1, 0)$ has norm 1 so there exists some $\varphi$ s.t. $R_2(1, 0) = (\cos\varphi, \sin\varphi)$. Again $R_2 \mapsto \varphi \in [-\pi, \pi]/(-\pi \sim \pi) = \mathbb{S}^1$ is continuous. Write $R_2(0, 1) = (-\sin\theta, \cos\theta)$ for some $\theta \in [-\pi, \pi]/(-\pi \sim \pi) = \mathbb{S}^1$. Then

$$1 = \det R_2 = \cos\varphi\cos\theta + \sin\varphi\sin\theta = \cos(\varphi - \theta).$$

Since $|\varphi - \theta| < 2\pi$ we conclude that $\theta = \varphi$. Hence $R_2 \mapsto \varphi$ is a bijection and R is a rotation around $\widehat{n}$ by an angle of $\varphi$. The map $R \mapsto \frac{\varphi}{\pi}\widehat{n}$ is thus a continuous bijection $SO(3) \to \overline{B_{\mathbb{R}^3}}(0, 1)/\sim$.

The inverse map is seen to be [8, p. 45]

$$\bar{\varphi} = \varphi\widehat{n} \mapsto R, \quad R_{jk} = \delta_{jk}\cos\varphi + n_j n_k(1 - \cos\varphi) - \sum_{m=1}^{3} \varepsilon_{jkm} n_m \sin\varphi.$$

We will now work in this set instead.

### Technical constructions

**Claim 1.** The space $\overline{B_{\mathbb{R}^3}}(0, 1)/\sim$ is a metric space.

*Proof of claim.* For a general metric space $(M, d_M)$ we may define a pseudometric on the quotient space $M/\sim$ by

$$d([x], [y]) = \inf\{d_M(p_1, q_1) + \ldots + d_M(p_n, q_n)\}.$$

where the infimum is taken over all finite paths from $[x]$ to $[y]$, such paths are given by $n \in \mathbb{N}, [p_1] = [x], [q_n] = [y], [p_{i+1}] = [q_i], i = 1, 2, \ldots, n - 1$. It is straightforward to prove that this defines a pseudometric.

For our case it is easy to show that this defines a metric and reduces to

$$d(x, y) = \min\left\{\inf\{d_{\mathbb{R}^3}(x, \nu) + d_{\mathbb{R}^3}(-\nu, y) : \nu \in \partial B_{\mathbb{R}^3}(0, 1)\}, \quad d_{\mathbb{R}^3}(x, y)\right\}. \qquad \circ$$

Note that for $d_{\mathbb{R}^3}(x, y)$ sufficiently small (say $< 1/2$) this metric agrees with the $\mathbb{R}^3$-metric, $d_{\mathbb{R}^3}(x, y) = d(x, y)$.

We will view $B_{\mathbb{R}^3}(0, 1) = B(0, 1) \subset \overline{B_{\mathbb{R}^3}}(0, 1)/\sim$ as a subset.

Suppose we have a homotopy $\gamma$ such that $\gamma(0, t) = 0 = \gamma(1, t)$, $\gamma(s, 0) = \bar{\gamma}(s)$, and $\gamma(s, 1) = 0$, where $\bar{\gamma}$ is some curve, to be specified later.

Since $\gamma$ is now a continuous function from a compact metric space $([0, 1] \times [0, 1])$ into another metric space we get that it is uniformly continuous.

Let $\varepsilon < \frac{1}{17}$. By uniform continuity find $\delta > 0$ such that

$$d(\gamma(s, t), \gamma(s', t')) < \varepsilon \quad \text{for } \|(s, t) - (s', t')\| < \delta.$$

Define a finite sequence

$$0 = s_0 < s_1 < \ldots < s_{N-1} < s_N = 1$$

such that $s_{k+1} - s_k < \delta$ for all k. Define for all t the sets

$$B_k^t = B(\gamma(s_k, t), \varepsilon) \cap B_{\mathbb{R}^3}(0, 1).$$

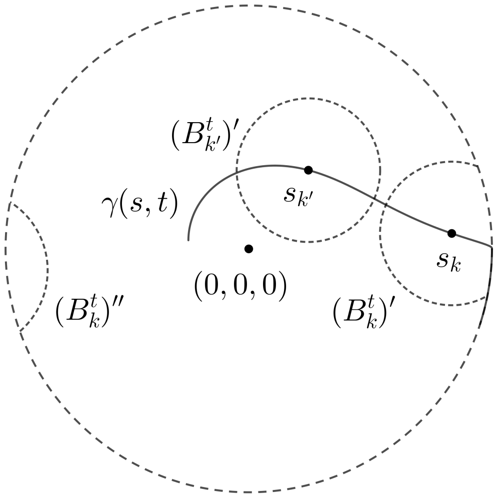

Figure 2: The two different cases of $B^t_k$. These sets have either 1 or 2 components. The points on the curve $s \mapsto \gamma(s,t)$ are labeled by their $s$-coordinate. For $k,t$ such that $\gamma(s_k,t)$ is close to the boundary, the balls $B(\gamma(s_k,t),\varepsilon)$ overlaps with the boundary $\partial B(0,1)$ and hence when removing this part, we get two disjoint components.

That is $B^t_k$ is the part of the $\varepsilon$-balls located in the open unit ball $B^t_k \subset B(0,1)$.

For each $t$ we will split up $B^t_k$ as a union of its connected components. Suppose first that $B^t_k$ is connected. Then we set $(B^t_k)' = B^t_k$ and $(B^t_k)'' = \varnothing$. If $B^t_k$ is not connected it has 2 components, call these $(B^t_k)'$ and $(B^t_k)''$, see figure 2 We want to label $(B^t_k)'$ and $(B^t_k)''$ as $B^t_{k,1}$ and $B^t_{k,2}$ according to where the curve $s \mapsto \gamma(s,t)$ lives. For the first, define

$$B^t_{0,1} = B(0,\varepsilon) = B^t_0 \text{ and } B^t_{0,2} = \varnothing.$$

Define $B^t_{k,1}$ and $B^t_{k,2}$ inductively as follows, see figure 3.

1. If $(B^t_{k+1})' \cap B^t_{k,1} \neq \varnothing$, then $B^t_{k+1,1} = (B^t_{k+1})'$ and $B^t_{k+1,2} = (B^t_{k+1})''$.
2. If $(B^t_{k+1})'' \cap B^t_{k,1} \neq \varnothing$, then $B^t_{k+1,1} = (B^t_{k+1})''$ and $B^t_{k+1,2} = (B^t_{k+1})'$.
3. If $(B^t_{k+1})' \cap B^t_{k,2} \neq \varnothing$, then $B^t_{k+1,1} = (B^t_{k+1})''$ and $B^t_{k+1,2} = (B^t_{k+1})'$.
4. If $(B^t_{k+1})'' \cap B^t_{k,2} \neq \varnothing$, then $B^t_{k+1,1} = (B^t_{k+1})'$ and $B^t_{k+1,2} = (B^t_{k+1})''$.

Note that we may be in cases 1 and 4 or cases 2 and 3 at the same time. This is not a problem as they agree on the definition of $B^t_{k+1,1}$ and $B^t_{k+1,2}$.

To prove that we may not be in both cases 1 and 3 (or similar) consider the $\mathbb{R}^3$ distance between the sets. By picking some antinodal points on the boundary and using the reverse triangle inequality it follows that $d_{\mathbb{R}^3}(B^t_{k,1}, B^t_{k,2}) \geq 2 - 4\varepsilon > 1$. Also $\mathrm{diam}_{\mathbb{R}^3}((B^t_{k+1})') \leq 2\varepsilon < 1$. Hence both intersections $(B^t_{k+1})' \cap B^t_{k,1}$ and $(B^t_{k+1})' \cap B^t_{k,2}$ can't be non-empty.

This defines for all $t$ the sets

$$B^t_{0,1}, B^t_{0,2}, B^t_{1,1}, \dots, B^t_{N,1}, B^t_{N,2}.$$

**Claim 2.** Let $i \in \{1,2\}$. If $\gamma(s_0,t_0) \in B^t_{k,i}$ then $\gamma(s_0, t_0+\rho) \in B^{t_0+\rho}_{k,i}$ for all $|\rho| < \rho_0$ for some $\rho_0 > 0$.
The intuition here is that the "1-sets" are close (in $\mathbb{R}^3$) and that the "2-sets" are close (in $\mathbb{R}^3$), but "1-sets" and "2-sets" are not (in $\mathbb{R}^3$), see figure 4.

*Proof of claim.* We use induction on $k$. We may wlog assume that $i = 1$. The result is clear for $k = 0$ so assume it hold for $k$.

Note first that if $d_{\mathbb{R}^3}(B^t_{k,1}, B^{t'}_{k',1}) \leq d$ then $d_{\mathbb{R}^3}(B^t_{k,1}, B^{t'}_{k',1}) \leq d + 4\varepsilon$. (This of course hold if we interchange 1 and 2 as well.) To see this use that $d = d_{\mathbb{R}^3}$ if the

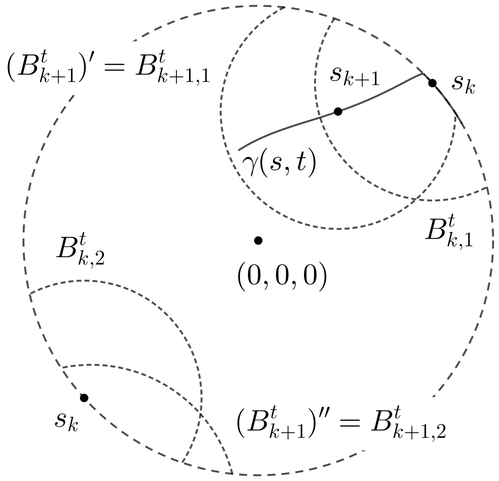

Figure 3: Construction of $B^t_{k,i}$. Here $s_k$ and $s_{k+1}$ label the points $\gamma(s_k,t)$ and $\gamma(s_{k+1},t)$ respectively. Note that we are in both cases 1 and 4 in this picture. The two points labeled $s_k$ are in fact the same point, as they are identified by the equivalence relation.

distance is small in $\mathbb{R}^3$ and find other points in the balls $B^t_k, B^{t'}_{k'}$ in the 2-component. An application of triangle inequality gives the desired.

Note also that $d_{\mathbb{R}^3}(B^t_{k,1}, B^t_{k+1,1}) \leq 4\varepsilon$ as we may just use the triangle inequality with inserting the centers of the circles.

If $\gamma(s,t) \in B^t_{k,1}$ for some $s$, then we may use the induction hypothesis to conclude that $\gamma(s, t+\rho) \in B^{t+\rho}_{k,1}$ for all $|\rho| < \rho_0$. Since moreover $\gamma$ is continuous and $d$ and the $\mathbb{R}^3$-metric agrees close to $\gamma(s,t) \in B(0,1)$, the interior, we see that $\gamma$ is continuous at the point $(s,t)$ w.r.t. the $\mathbb{R}^3$ metric. Hence we conclude that $d_{\mathbb{R}^3}(\gamma(s,t), \gamma(s,t+\rho)) \leq \varepsilon$ for all $|\rho| < \rho_0$ for some possibly smaller $\rho_0$. In this case then

$$d_{\mathbb{R}^3}(B^t_{k,1}, B^{t+\rho}_{k,1}) \leq \varepsilon.$$

If instead $\gamma(s,t) \in B^t_{k,2}$ for some $s$ we may similarly conclude that

$$d_{\mathbb{R}^3}(B^t_{k,2}, B^{t+\rho}_{k,2}) \leq \varepsilon.$$

Using the above comment we get that $d_{\mathbb{R}^3}(B^t_{k,1}, B^{t+\rho}_{k,1}) \leq 5\varepsilon$.

The last case is that $\gamma(s,t) \in \partial B(0,1)$ for all $s$ "close to" $s_k$. Then in fact the sets $B^t_{k,1}$ overlap and thus

$$d_{\mathbb{R}^3}(B^t_{k,1}, B^{t+\rho}_{k,1}) = 0.$$

Consider now $x \in B^t_{k+1,1}$ and $y \in B^{t+\rho}_{k+1,1}$. Let $x_0 \in B^t_{k,1}, y_0 \in B^{t+\rho}_{k,1}$. Then by combining the above we have the bound

$$d_{\mathbb{R}^3}(x,y) \leq d_{\mathbb{R}^3}(x,x_0) + d_{\mathbb{R}^3}(x_0,y_0) + d_{\mathbb{R}^3}(y_0,y) \leq 4\varepsilon + 5\varepsilon + 4\varepsilon = 13\varepsilon$$

We have some problems if some of these sets are empty. In this case we may apply the above bound on the "2-sets" and thus by the begining comments bound the $\mathbb{R}^3$ distance by

$$d_{\mathbb{R}^3}(x,y) \leq 13\varepsilon + 4\varepsilon = 17\varepsilon.$$

In total we have the bound (if $B^t_{k+1,1} \neq \varnothing$)

$$d_{\mathbb{R}^3}(B^t_{k+1,1}, B^{t+\rho}_{k+1,1}) \leq 17\varepsilon.$$

Suppose now $\gamma(s_0,t_0) \in B^t_{k+1,1}$. Again using that $\gamma$ is continuous we may conclude that $\gamma(s_0, t_0+\rho) \in B(0,1)$ for all $|\rho| < \rho_0$. (For a possibly smaller $\rho_0$.) We conclude that $\gamma(s_0, t_0+\rho) \in B^{t+\rho}_{k,j}$ for some $j \in \{1,2\}$. The above calculations show that $d_{\mathbb{R}^3}(B^t_{k,1}, B^{t+\rho}_{k,1}) \leq 17\varepsilon$ and thus that

$$d_{\mathbb{R}^3}(B^t_{k+1,1}, B^{t+\rho}_{k+1,2}) \geq 2 - 17 > 1$$

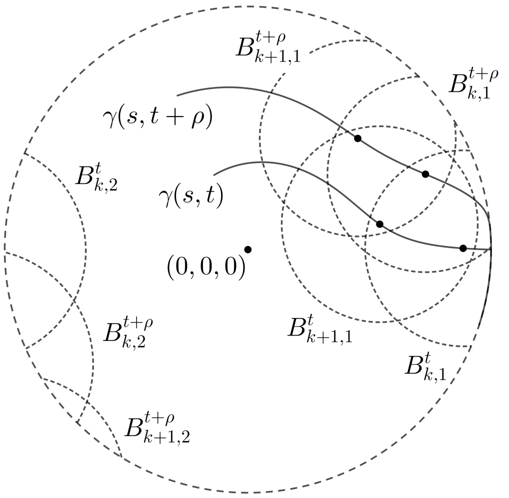

Figure 4: Picture of the $t+\rho$ sets being close. We have a lot of overlap of the "1-sets", where this particular curve, $s \mapsto \gamma(s,t)$, lives. Note that even though $B^{t}_{k+1,1}$ and $B^{t+\rho}_{k+1,1}$ overlap, we do not have that $B^{t}_{k+1,2}$ and $B^{t+\rho}_{k+1,2}$ overlap. In fact $B^{t}_{k+1,2} = \varnothing$.

We conclude that $j \neq 2$ and thus that $\gamma(s_0, t_0+\rho) \in B^{t_0+\rho}_{k+1,1}$ for all $|\rho| < \rho_0$.

This concludes the induction step and thus we have proved the claim. ○

Define now $\kappa(s) = \sup\{k : s_k \leq s\}$ and for $t \in B$

$$s^{t}_{\text{final}} = \sup\{s : \|\gamma(s,t)\| = 1\}, \qquad \sigma^t = \frac{s^{t}_{\text{final}} + s_{\kappa(s^{t}_{\text{final}})+1}}{2}$$

As the last time the curve $s \mapsto \gamma(s,t)$ hits the boundary and some appropriately small time after that. Then

$$s_{\kappa(s^{t}_{\text{final}})} \leq s^{t}_{\text{final}} < \sigma^t < s_{\kappa(s^{t}_{\text{final}})+1}$$

Hence $\kappa(s^{t}_{\text{final}}) = \kappa(\sigma^t)$ and so $\gamma(\sigma^t, t) \in B^{t}_{\kappa(\sigma^t)}$, see figure 5. We say that $\gamma$ *jumps at time* $t$ if $\gamma(\sigma^t, t) \in B^{t}_{\kappa(\sigma^t),2}$. (And thus that $\gamma$ does not jump if $\gamma(\sigma^t,t) \in B^{t}_{\kappa(\sigma^t),1}$.)

Define the sets

$$\begin{aligned} I &= \{t : \|\gamma(s,t)\| < 1 \text{ for all } 0 \leq s \leq 1\}, \\ B &= \{t : \|\gamma(s,t)\| = 1 \text{ for some } 0 \leq s \leq 1\}, \\ J &= \{t \in B : \gamma \text{ jumps at time } t\}. \end{aligned}$$

Note that since at $t = 1$ the curve is constant we have that $1 \in I$.

### "Jumps can't disappear"

**Claim 3.** I is open

The idea for the proof of this claim was given to me by my advisor Jan Philip Solovej.

*Proof of claim.* We prove that $f : t \mapsto \sup_{s \in [0,1]}\{\|\gamma(s,t)\|\}$ is continuous.

Let $t \in [0,1]$ and let $t^j \to t$ be a sequence converging to $t$. By compactness the supremum is attained so $f(t^j) = \left\|\gamma(s^j, t^j)\right\|$ for some $s^j$. By compactness we may assume (by passing to a subsequence) that $s^j \to s$ for some $s \in [0,1]$. Now the limit is $\lim_{j\to\infty} f(t^j) = \|\gamma(s,t)\| \leq f(t)$. Suppose for contradiction that $f(t) > \|\gamma(s,t)\|$. Then again $f(t) = \|\gamma(s_0,t)\|$ for some $s_0$. Then

$$\lim_{j\to\infty} \left\|\gamma(s^j, t^j)\right\| = \|\gamma(s,t)\| < \|\gamma(s_0,t)\| = \lim_{j\to\infty} \left\|\gamma(s_0, t^j)\right\|.$$

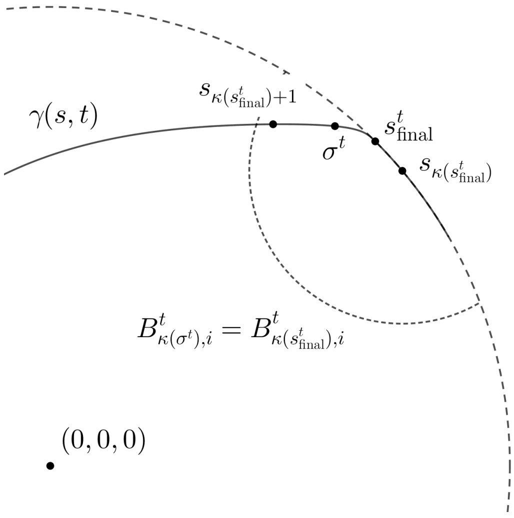

Figure 5: Picture of the curve "leaving the boundary". The different points $\gamma(s_{\kappa(s^t_{\mathrm{final}})}, t), \gamma(s^t_{\mathrm{final}}, t), \gamma(\sigma^t, t), \gamma(s_{\kappa(s^t_{\mathrm{final}})+1}, t)$ are labeled by their $s$-coordinate. $i \in \{1,2\}$. Note that only the points $\gamma(\sigma^t, t)$ and $\gamma(s_{\kappa(s^t_{\mathrm{final}})+1}, t)$ lie in the interior $B(0,1)$. The remaining points lie on the boundary $\partial B(0,1)$. In general $\gamma(s_{\kappa(s^t_{\mathrm{final}})+1}, t)$ need not lie on the boundary.

Hence for $j > j_0$ for some $j_0$ we have that $\|\gamma(s^j, t^j)\| < \|\gamma(s_0, t^j)\|$. But $s^j$ was by construction a maximizer. Contradiction. We conclude that $\lim_{j\to\infty} f(t^j) = f(t)$ and thus that $f$ is continuous. Now $I$ is the preimage of the open set $[0,1)$ under $f$ hence open. (Note that we work in the standard topology on $[0,1]$.) $\circ$

**Claim 4.** $\bar{I} \cap J = \varnothing$

*Proof of claim.* Suppose for contradiction that $t \in \bar{I} \cap J \neq \varnothing$.

Let $(t^i) \subset I$ be a sequence converging to $t$.

Since $\|\gamma(s, t^i)\| < 1$ for all $s$ we have that $\gamma(s, t^i) \in B^{t^i}_{\kappa(s),1}$ for all $s$. In particular $\gamma(\sigma^t, t^i) \in B^{t^i}_{\kappa(\sigma^t),1}$. On the other hand we have that $\gamma(\sigma^t, t) \in B^t_{\kappa(\sigma^t),2}$ by assumption of $t \in J$ .

By continuity we conclude that there exists some $i_0$ s.t. $\gamma(\sigma^t, t^i) \in B^t_{\kappa(\sigma^t),2}$ for all $i \geq i_0$. Since also $\gamma(\sigma^t, t^i) \in B^{t^i}_{\kappa(\sigma^t),1}$ we conclude that $B^{t^i}_{\kappa(\sigma^t),1} \cap B^t_{\kappa(\sigma^t),2} \neq \varnothing$ for all $i \geq i_0$. But in general we have that $B^{t_1}_{k,1} \cap B^{t_2}_{k,2} = \varnothing$ for $|t_2 - t_1|$ small enough. Contradiction. We conclude that $\bar{I} \cap J = \varnothing$. $\circ$

**Claim 5.** $J$ is open.

*Proof of claim.* Let $t \in J$. We prove that for some $\rho_0 > 0$, $t + \rho \in J$ for all $|\rho| < \rho_0$.

Now, $t \notin \bar{I}$ so $t \in B \backslash \partial B$ so $t + \rho \in B$ for all $|\rho| < \rho_0$ for some $\rho_0 > 0$.

By assumption $\gamma(\sigma^t, t) \in B^t_{\kappa(\sigma^t),2}$.

By claim 2 we have that $\gamma(\sigma^t, t+\rho) \in B^{t+\rho}_{\kappa(\sigma^t),2}$ for all $|\rho| < \rho_0$ for some possibly smaller $\rho_0 > 0$.

Now, $\sigma^{t+\rho} - \sigma^t$ may not be small. Hence we may not just do a continuity argument here. We may deal with the $\rho$ for which $\sigma^{t+\rho} > \sigma^t$ in the following way.

Restricting $s$ to the set $[\sigma^t, 1]$ we see that at time $t$ the curve does not touch the boundary in this interval. Now the set

$$I_{[\sigma^t,1]} = \{t' \in [0,1] : \|\gamma(s,t')\| < 1 \text{ for all } \sigma^t \leq s \leq 1\}$$

may be shown to be open analogously to $I$. Note that $t \in I_{[\sigma^t,1]}$. Hence for some possibly smaller $\rho_0 > 0$ we have that $t + \rho \in I_{[\sigma^t,1]}$ for all $|\rho| < \rho_0$. So the case $\sigma^{t+\rho} > \sigma^t$ does not happen.

Assume now $\sigma^{t+\rho} \leq \sigma^t$. Since $\|\gamma(s, t+\rho)\| < 1$ for $s \in [\sigma^{t+\rho}, \sigma^t]$ we have that the curve can't jump between "1-sets" and "2-sets" in this interval. We know that $\gamma(\sigma^t, t+\rho) \in B^t_{\kappa(\sigma^t),2}$ we see that $\gamma(\sigma^{t+\rho}, t+\rho) \in B^t_{\kappa(\sigma^{t+\rho}),2}$. We conclude that the curve jumps at time $t+\rho$ and thus that $J$ is open. ◦

**Claim 6.** J is closed

*Proof of claim.* Note that $\partial J \subset B$ as $B$ is closed. Moreover, since $J$ is open we have that $\partial J \subset B \backslash J$.

We may describe $B \backslash J$ as the set of $t$ for which $\gamma$ does not jump. But a better way is to see it as $t \in B$ for which

$$\gamma(\sigma^t, t) \in B^t_{\kappa(\sigma^t),1}$$

This is (almost) the same as that of $J$, only with index 1 instead of 2. Hence we may do the same argument as in claim 5. The problem is that we may not conclude that $t+\rho \in B$ for all $|\rho| < \rho_0$, since we may have $t \in \partial B = \partial I$. But we may still conclude that, if $t+\rho \in B$ for some $|\rho| < \rho_0$, then $\gamma$ does not jump at time $t+\rho$. We may fix this problem of not being an element of $B$ for certain $t$ by restricting to a subsequence.

Suppose now for contradiction $\partial J \neq \varnothing$ and let $t \in \partial J \subset B \backslash J$. Find a sequence $(t^j) \subset J \subset B$ s.t. $t^j \to t$. Then $|t^j - t| < \rho_0$ for all $j \geq j_0$ for some $j_0 \in \mathbb{N}$. Then $t^{j_0} = t + \rho$ for some $|\rho| < \rho_0$ and $t^{j_0} \in J \subset B$ so we may conclude as above that $\gamma$ does not jump at time $t^{j_0}$, i.e. that $t^{j_0} \notin J$. But by construction $t^{j_0} \in J$. Contradiction. We conclude that $\partial J = \varnothing$ and thus $J = J \cup \partial J = \overline{J}$ is closed. ◦

In total: J is both open and closed.

Define the curve

$$\bar{\gamma}(s) = \begin{cases} (2s, 0, 0) & \text{if } 0 \leq s \leq \frac{1}{2} \\ (2s-2, 0, 0) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

We see that this curve jumps, hence $\gamma$ has a jump at $t = 0$, i.e. $0 \in J$. We conclude that $J = [0, 1]$ by connectedness. On the other hand $1 \in I$ so $1 \notin J$. Contradiction. Hence $\bar{\gamma}$ is not a contractible curve.

We conclude that $SO(3)$ is not simply connected.

# 3 Lifting Projective Representations

In this section we construct a bijection between the projective representations of the Poincaré group and the representations of its covering group. This will help us classify the projective representations of the Poincaré group. We first prove the existence of such a bijection in the general setting under some conditions, which we subsequently prove hold for the Poincaré group. This section is based on [8, p. 49, 63-66].

## 3.1 The Covering Group

We find general conditions on a Lie group such that we may construct a bijection between projective unitary representations of the group and unitary representation of the covering group.

**Definition 3.1.** Let $\mathcal{G}$ be a Lie group.

A Lie group $\tilde{\mathcal{G}}$ is said to be the covering group of $\mathcal{G}$ if $\tilde{\mathcal{G}}$ is simply connected and there exists a surjective open continuous group homomorphism $\tilde{p} : \tilde{\mathcal{G}} \to \mathcal{G}$.

We may then write (with $\mathcal{H} = \ker \tilde{p}$) $\mathcal{G} \cong \tilde{\mathcal{G}} / \ker \tilde{p} = \tilde{\mathcal{G}} / \mathcal{H}$.

Note that that the usual definition requires the following additional condition as well, see [5, p. 332, 450].

For all $g \in \mathcal{G}$ there exists some open set $U \subset \mathcal{G}$, with $g \in U$ and $p^{-1}(U) = \bigcup_\alpha V_\alpha$ is a disjoint union of open sets $V_\alpha \subset \tilde{G}$ such that $\tilde{p}|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism.

This additional property is clear for the covering groups, which we consider, and for all the proofs we need only the properties stated in our definition. We will thus ignore this additional condition.

**Example 3.2.** A well-known example of a covering group, is that $\widetilde{SO(3)} = SU(2)$. This follows from Prop. 3.10.

**Theorem 3.3.** *Let $\mathcal{G}$ be a connected Lie group and $\tilde{\mathcal{G}}$ be a covering group with covering map $\tilde{p} : \tilde{\mathcal{G}} \to \mathcal{G}$. Write $\mathcal{G} \cong \tilde{\mathcal{G}}/\mathcal{H}$, with $\mathcal{H} = \ker \tilde{p}$.*

*Suppose that every projective unitary representation, $\rho' : \tilde{\mathcal{G}} \to U(\mathfrak{H})$ of the covering group admits a lifting, $\rho' = p \circ \pi$, where $\pi$ is a unitary representation of $\tilde{\mathcal{G}}$.*

*Then we have a bijective correspondence*

$$\left\{ \begin{array}{c} \rho : \mathcal{G} \to U(\widehat{\mathfrak{H}}) \\ \text{projective unitary} \\ \text{representation of } \mathcal{G} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \pi : \tilde{\mathcal{G}} \to U(\mathfrak{H}) \\ \text{unitary representation of } \tilde{\mathcal{G}} \\ \text{such that } \pi(\mathcal{H}) \subset \ker p \end{array} \right\}$$

*Proof.* Let $\pi : \tilde{\mathcal{G}} \to U(\mathfrak{H})$ be a unitary representation s.t. $\pi(\mathfrak{H}) \subset \ker p$. Now the map $p \circ \pi$ is trivial on $\mathcal{H}$ so by the universal property of the projection $\tilde{p}$ we have the following diagram

$$\begin{tikzcd} \tilde{\mathcal{G}} \arrow[r, "\pi"] \arrow[d, "\tilde{p}"'] & U(\mathfrak{H}) \arrow[d, "p"] \\ G \arrow[r, dashed, "\exists!\rho"] & U(\widehat{\mathfrak{H}}) \end{tikzcd}$$

There exist a unique map $\rho$ such that this commutes. Now continuity of $\rho$ follows from the universal property of quotient maps ($\tilde{p}$ is a quotient map, as it is open, continuous and surjective), that $\rho$ is a group homomorphism follows from the universal property of projections in the setting of groups. We conclude that $\rho$ is a projective unitary representation.

By assumption every $\rho$ arises this way: Given $\rho$ we may define $\rho' = \rho \circ \tilde{p}$ and use the assumption to get a unitary representation $\pi$. Doing the above construction we regain $\rho$, since $\rho$ makes the diagram commute, and this property identifies $\rho$ uniquely. ☐

**Proposition 3.4.** In the bijective correspondence above ($\rho \mapsto \pi$) we have that $\pi$ is irreducible if and only if $\rho$ is irreducible.

*Proof.* Same as that of Prop. 2.26. ☐

We now give a criterion for when a Lie group has such a lifting property.

**Definition 3.5.** A Lie algebra $\mathfrak{g}$ is said to have trivial second cohomology group if for any bilinear map $\theta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ satisfying

$$\theta(\mathbf{A}, \mathbf{B}) = -\theta(\mathbf{B}, \mathbf{A}),$$
$$\theta([\mathbf{A}, \mathbf{B}], \mathbf{C}) + \theta([\mathbf{B}, \mathbf{C}], \mathbf{A}) + \theta([\mathbf{C}, \mathbf{A}], \mathbf{B}) = 0$$

for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathfrak{g}$. Then there exist some linear map $\vartheta : \mathfrak{g} \to \mathbb{R}$ s.t.

$$\theta(\mathbf{A}, \mathbf{B}) = \vartheta([\mathbf{A}, \mathbf{B}])$$

for all $\mathbf{A}, \mathbf{B} \in \mathfrak{g}$.

**Theorem 3.6** (Bargmann)**.** *Let $\mathcal{G}$ be a connected and simply connected Lie group. Let $L_\mathcal{G}$ denote its Lie algebra. Suppose that $L_\mathcal{G}$ has trivial second cohomology group. Then any projective unitary representation admits a lifting.*

We will not prove this result but refer to [6, p. 9-16] for the details.

## 3.2 The Covering Group of the Poincaré Group

The previous section motivates that we should prove that the Lie algebra of the covering group of the Poincaré group has trivial second cohomology group. We start by finding the covering group.

The first result is that

**Theorem 3.7.** *The covering group of the restricted Lorentz group is $\tilde{\mathcal{L}}_+^\uparrow = SL_2(\mathbb{C})$.*

To prove this we do the following

Let $H(2) = \{\sigma \in SL_2(\mathbb{C}) | \overline{\sigma} = \sigma^T\}$ be the set of all Hermitian $2 \times 2$ matrices. Now $H(2)$ is a vectorspace over $\mathbb{R}$ with basis

$$\left\{ \sigma_0 = I_2, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

This is seen as for any $\sigma \in H(2)$ we have $\sigma_{12} = \overline{\sigma_{21}}$ and $\sigma_{11}, \sigma_{22} \in \mathbb{R}$ so

$$\sigma = \begin{bmatrix} a & \overline{c} \\ c & b \end{bmatrix} = \frac{a+b}{2}\sigma_0 + \mathfrak{R}(c)\sigma_1 + \mathfrak{I}(c)\sigma_2 + \frac{a-b}{2}\sigma_3.$$

Define $\phi : \mathbb{R}^4 \to H(2)$ by $\phi(x) = \sum_\mu x^\mu \sigma_\mu$. This is an isomorphism of vectorspaces over $\mathbb{R}$. Now any matrix $A \in SL_2(\mathbb{C})$ induces a linear map of $\mathbb{R}^4$ by the following diagram

$$\begin{tikzcd}
H(2) \arrow[r, "A(-)A^*"] & H(2) \\
\mathbb{R}^4 \arrow[u, "\phi", "\cong"'] \arrow[r, dashed, "\Lambda_A"] & \mathbb{R}^4 \arrow[u, "\phi", "\cong"']
\end{tikzcd}$$

That is $\Lambda_A x = \phi^{-1}(A\phi(x)A^*)$. (Note that the map $A(-)A^*$ is well-defined.)

**Lemma 3.8.** The map $\Lambda : SL_2(\mathbb{C}) \to \mathcal{L}_+^\uparrow, A \mapsto \Lambda_A$ is an open surjective continuous group homomorphism with kernel $\ker \Lambda = \{\pm I_2\}$.

*Proof.* The first thing to show is that for $SL_2(\mathbb{C})$ matrices $A$, we have that $\Lambda_A$ is a Lorentz transformation.

We have that $\langle x, x \rangle = \det \phi(x)$ so we see that

$$\langle \Lambda_A x, \Lambda_A x \rangle = \det \phi(\Lambda_A x) = \det A \det \phi(x) \det A^* = \langle x, x \rangle.$$

By the parallelogram law we thus get that

$$\begin{aligned} \langle \Lambda_A x, \Lambda_A y \rangle &= \frac{\langle \Lambda_A(x+y), \Lambda_A(x+y) \rangle - \langle \Lambda_A(x-y), \Lambda_A(x-y) \rangle}{4} \\ &= \frac{\langle x+y, x+y \rangle - \langle x-y, x-y \rangle}{4} = \langle x, y \rangle \end{aligned}$$

so $\Lambda_A$ is a Lorentz transformation. The things left to prove is that $(\Lambda_A)^0{}_0 \geq 1$ and that $\det \Lambda_A = 1$. For the first of these consider the images of $e_0 = (1, 0, 0, 0)$ in the diagram.

$$\begin{tikzcd}
I_2 \in H(2) \arrow[r, "A(-)A^*"] & H(2) \ni AA^* \\
e_0 \in \mathbb{R}^4 \arrow[u, "\phi", "\cong"'] \arrow[r, "\Lambda_A"] & \mathbb{R}^4 \ni \Lambda_A e_0 \arrow[u, "\phi", "\cong"']
\end{tikzcd}$$

Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$A\sigma_0 A^* = AA^* = \begin{bmatrix} |a|^2 + |b|^2 & a\overline{c} + b\overline{d} \\ \overline{a}c + \overline{b}d & |c|^2 + |d|^2 \end{bmatrix}$$

And by the description above we see that

$$(\Lambda_A)^0{}_0 = (\Lambda_A e_0)^0 = \frac{|a|^2 + |b|^2 + |c|^2 + |d|^2}{2} > 0.$$

Since $\Lambda_A$ is a Lorentz transformation we know that $|(\Lambda_A)^0{}_0| \geq 1$ (see section 2.1). We conclude that $(\Lambda_A)^0{}_0 \geq 1$.

For the second part, $\det \Lambda_A = 1$ we first prove the continuity.

The fact that $\Lambda$ is a group homomorphism is clear by construction. Openness and continuity of the map follows from the fact that all the coordinates of $\Lambda_A$ are polynomial in the coefficients of $A$.

The map $A \mapsto \Lambda_A$ is continuous, and since $SL_2(\mathbb{C})$ is connected by Prop. 3.9, we see that the image is connected. Since taking determinants is continuous we see that the set of determinants $\{\det \Lambda_A : A \in SL_2(\mathbb{C})\}$ is connected. Now, $\Lambda_A$ is a Lorentz transformation for all $A$ and so $\det \Lambda_A = \pm 1$. But clearly $\det \Lambda_{I_2} = 1$. We conclude that $\det \Lambda_A = 1$ for all $A \in SL_2(\mathbb{C})$.

We are left with proving surjectivity. We give an idea for the proof.

Write out an explicit formula for $\Lambda_A$ in terms of the coefficients of $A$. Then the equations $\Lambda = \Lambda_A$ is just some polynomial equations in the coefficients of $A$, solve them to get $A \in M_{2\times 2}(\mathbb{C})$. Going through the above calculations, we see that we need $|\det A| = 1$ for $\Lambda_A$ to be a Lorentz transformation. Hence we may pick $A' = \frac{1}{\sqrt{\det A}} A$ (with some choice of square root).

For the kernel we have

$$\ker \Lambda = \{A \in SL_2(\mathbb{C}) : A\sigma A^* = \sigma \text{ for all } \sigma \in H(2)\}.$$

By taking $\sigma = I_2$ in this we see that $A \in \ker \Lambda$ is unitary. So $A\sigma = \sigma A$. By taking $\sigma = \sigma_3$ we see that $A$ only has diagonal entries, and by further by taking $\sigma = \sigma_2$ we get that $A = \lambda I_2$ for some $\lambda \in \mathbb{C}$. In total we see, that

$$\ker \Lambda \subset \{\lambda I_2 : \lambda \in \mathbb{C}\} \cap U(2) = \{\pm I_2\}.$$

On the other hand $\{\pm I_2\} \subset \ker \Lambda$ clearly. ☐

To show that $SL_2(\mathbb{C})$ is the covering group we need the following result.

**Proposition 3.9.** $SL_2(\mathbb{C})$ is connected and simply connected.

The idea for this proof was given to me by my advisor Jan Philip Solovej.

*Proof.* We prove the homeomorphisms

$$SL_2(\mathbb{C}) \cong SU(2) \times M \cong \mathbb{S}^3 \times \mathbb{R}^3$$

where $M$ is the set $\{H \in SL_2(\mathbb{C}) : H^* = H, \text{tr } H > 0, \det H = 1\}$. Since the latter is connected and simply connected [5, p. 154, 365] we have the result.

The homeomorphism $SL_2(\mathbb{C}) \cong SU(2) \times M$ is what is normally called polar decomposition, see [8, p. 69]

Define first the map $SL_2(\mathbb{C}) \to SU(2) \times M$ by the following.

Let $A \in SL_2(\mathbb{C})$, then $(AA^*)^* = AA^*$ is hermitian. Hence we may diagonalize $AA^* = V^*DV$, for some $V$, $D = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}$, $\mu_1, \mu_2 \in \mathbb{C}$. We claim that $\mu_1, \mu_2 \geq 0$.

To prove this let $x \in \mathbb{C}^2$ be an eigenvector with eigenvalue $\mu_1$, then we have

$$\mu_1 \|x\|^2 = \langle \mu_1 x, x \rangle = \langle AA^* x, x \rangle = \langle A^* x, A^* x \rangle \geq 0$$

and similarly for $\mu_2$. Hence $(AA^*)^{1/2} = V^* D^{1/2} V$ is well-defined.

Note that $\det(AA^*) = |\det A|^2 = 1$ hence $(AA^*)^{1/2}$ is invertible and $\mu_1\mu_2 = 1$ so $\mu_1, \mu_2 > 0$.

Define now $U = (AA^*)^{-1/2} A$. Then we see that $U$ is unitary as follows

$$UU^* = (AA^*)^{-1/2} A \big((AA^*)^{-1/2} A\big)^* = (AA^*)^{-1/2} (AA^*) (AA^*)^{-1/2} = I_2.$$

Moreover, $\det U = 1$. Define $H = (AA^*)^{1/2} = AU^{-1}$. Then $A = HU$. Now $H$ satisfies $H^* = H$ and

$$\text{tr } H = \text{tr } (AA^*)^{1/2} = \text{tr } V^* D^{1/2} V = \text{tr } D^{1/2} = \sqrt{\mu_1} + \sqrt{\mu_2} > 0.$$

This gives the first map $SL_2(\mathbb{C}) \ni A \mapsto (U, H) \in SU(2) \times M$. Injectivity is clear as the (left) inverse is given by $(U, H) \mapsto HU$.

To prove that it is surjective note that $(HU)(HU)^* = H^2$, and since we only have one choice of square root $H'$ with both $\text{tr } H' > 0$ and $\det H' = 1$, we see that $H' = H$ and hence this map is surjective.

It is clear, the both the map and the inverse are continuous. This proves the first homeomorphism.

To see that $M \cong \mathbb{R}^3$, note that any $H \in M$ must (by a similar argument as for $H(2)$) be of the form

$$H = \begin{bmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{bmatrix}$$

with $0 < \operatorname{tr} H = 2x^0$, and $1 = \det H = (x^0)^2 - \bar{x}^2$. Hence $x^0 = \sqrt{1 + \bar{x}^2}$. It is clear that the map $H \mapsto \bar{x} \in \mathbb{R}^3$ is a homeomorphism.

To see that $SU(2) \cong \mathbb{S}^3$ note that

$$SU(2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \begin{matrix} ad - bc = 1, & |a|^2 + |b|^2 = 1 \\ |c|^2 + |d|^2 = 1, & a\bar{c} + b\bar{d} = 0 \end{matrix} \right\},$$

which is seen by writing out the components of $I_2 = UU^*$ for $U \in SU(2)$ and $1 = \det U$. These equations then give that

$$SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \middle| |a|^2 + |b|^2 = 1 \right\},$$

from which it follows, that

$$SU(2) \ni \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \mapsto \big(\mathfrak{R}(a), \mathfrak{I}(a), \mathfrak{R}(b), \mathfrak{I}(b)\big) \in \mathbb{S}^3$$

is a homeomorphism. □

In total we have

*Proof of Thm. 3.7.* Combine Lem. 3.8 and Prop. 3.9. □

Using this we see that $\tilde{\mathcal{P}}_+^\uparrow = \mathbb{R}^4 \rtimes \tilde{\mathcal{L}}_+^\uparrow = \mathbb{R}^4 \rtimes SL_2(\mathbb{C})$ is the covering group of the restricted Poincaré group.

Something to note here is which $A$ gets mapped to the rotations. These are exactly the one preserving $e_0$ i.e. $\sigma_0 = I_2$. That is $A\sigma_0 A^* = \sigma_0$. Hence $AA^* = I_2$ So these are unitary. We conclude that the rotations are exactly $SU(2)$. Note that this also proves the following.

**Proposition 3.10.** The covering group of $SO(3)$ is $\widetilde{SO(3)} = SU(2)$.

The decomposition of $A$ into $HU$ above then correspond to how any Lorentz transformation may be written as a product of a boost and a rotation.

We now want to find the covering group of $\mathcal{P}^\uparrow$. Again we first find the covering group of $\mathcal{L}^\uparrow$, as the extension to $\mathcal{P}^\uparrow$ is easy. This will amount to adding an element $L_P$ to $\tilde{\mathcal{L}}_+^\uparrow$ such that $L_P \mapsto P$.

For matrices $A \in SL_2(\mathbb{C})$ define

$$L_A = \begin{bmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{bmatrix} \quad \text{and} \quad L_P = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}.$$

**Theorem 3.11.** *The covering group of the orthochronous Lorentz group is given by* $\tilde{\mathcal{L}}^\uparrow = \{L_A, L_P L_A : A \in SL_2(\mathbb{C})\}$. *The covering map is given by*

$$\begin{aligned} L_A &\mapsto \Lambda_A \\ L_P &\mapsto P \end{aligned}$$

*Proof.* We need to show that this map is well-defined. This amounts to showing that if $L_P L_A = L_{A'} L_P$ then $P\Lambda_A = \Lambda_{A'} P$. The commutation relation between $L_P$ and the $L_A$'s is given by $L_{(A^*)^{-1}} = L_P L_A L_P^{-1}$ as is an easy computation. We need to prove that

$$\Lambda_{(A^*)^{-1}} = P\Lambda_A P^{-1}.$$

To prove this consider the diagram

$$\begin{tikzcd}
H(2) \arrow[rrr, bend left, "(A^*)^{-1}(-)A^{-1}"] & H(2) \arrow[l, "(-)^{-1}\,\mathtt{det}(-)"'] \arrow[r, "A(-)A^*"'] & H(2) \arrow[r, "(-)^{-1}\,\mathtt{det}(-)"'] & H(2) \\
\mathbb{R}^4 \arrow[u, "\phi"] \arrow[u, "\cong"'] \arrow[rrr, bend right, "P\Lambda_A P^{-1}"'] & \mathbb{R}^4 \arrow[l, "P"'] \arrow[r, "\Lambda_A"] \arrow[u, "\phi"] \arrow[u, "\cong"'] & \mathbb{R}^4 \arrow[r, "P"] \arrow[u, "\phi"] \arrow[u, "\cong"'] & \mathbb{R}^4 \arrow[u, "\phi"] \arrow[u, "\cong"']
\end{tikzcd}$$

The left (and right) squares commutes by a straightforward computation. (Note that all the maps are linear, so it suffices to check commutativity on a basis.) The top commutes since $\sigma_0^2 = \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I_2$. In conclusion the outer square commutes. This proves that the map is well-defined and a group homomorphism (since $A \mapsto \Lambda_A$ is). Since $A \mapsto \Lambda_A$ is open continuous and surjective we get that the above map is as well.

It remains to show that $\{L_A, L_P L_A : A \in SL_2(\mathbb{C})\}$ is simply connected.

Multiplication by $L_P$ is a homeomorphism, as it is its own inverse, and multiplication is continuous. Hence $\{L_A, L_P L_A : A \in SL_2(\mathbb{C})\}$ is the disjoint union of the two subsets $\{L_A : A \in SL_2(\mathbb{C})\}$ and $\{L_P L_A : A \in SL_2(\mathbb{C})\}$, which, both being homeomorphic to $SL_2(\mathbb{C})$, are simply connected. The set $\{L_A, L_P L_A : A \in SL_2(\mathbb{C})\}$ is thus simply connected. We conclude that $\tilde{\mathcal{L}}^{\uparrow} = \{L_A, L_P L_A : A \in SL_2(\mathbb{C})\}$ is the covering group. □

## 3.3 Lie Algebra of the Poincaré Group

We compute the Lie algebra of the (covering group of the) Poincaré group to see that is has trivial second cohomology group. The way, we compute it, (see below) is by using the commutation relations of the group. These are the same as that of the covering group, hence we may just consider the Lie algebra of the Poincaré group. To do this we will find the generators of one-parameter subgroups spanning the whole group.

First note that $(\mathbb{R}^n, +)$ has Lie algebra $\mathbb{R}^n$ with the exponential map $\exp : x \mapsto x$ [3]. This is seen by the group isomorphism

$$\mathbb{R}^n \ni (x^1, \dots, x^n) \mapsto \begin{bmatrix} e^{x^1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{x^n} \end{bmatrix} \in \left\{ \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \in GL_n(\mathbb{R}) \middle| \lambda_k > 0 \right\}$$

Any Lorentz transformation may by Prop. 2.9 be parameterized by 6 coordinates and the Poincaré group may then be parametrized by 10 coordinates. Moreover, the discussion just before Prop. 2.9 gives us one-parameter subgroups $t \mapsto \bar{\omega} = te_j$, $j = 1, 2, 3$, and $t \mapsto \bar{\varphi} = te_j$, $j = 1, 2, 3$. Combining with the 4 one-parameter subgroups describing the translations, $t \mapsto (te_\mu, I_4) \in \mathcal{P}^{\uparrow}$, $\mu = 0, 1, 2, 3$, we have in total 10 one-parameter subgroups spanning the entire group. We will denote the infinitesimal generators as follows

| | |
|---|---|
| $-\mathbf{H}_0$ | Translation in the $x^0$ direction, |
| $\mathbf{p}_j$ | Translation in the $x^j$ direction, $j = 1, 2, 3$, |
| $-\mathbf{N}_j$ | Boost in the $x^j$ direction, $j = 1, 2, 3$, |
| $\mathbf{J}_j$ | Rotation around/in the $x^j$ direction, $j = 1, 2, 3$. |

The last 6 of these are generators of a matrix Lie algebra, (in explicit matrix form) so we may calculate them as matrices and compute their commutation relations using these explicit descriptions. We have the explicit forms

$$\mathbf{N}_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{J}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

And similarly for the rest, see section A.3.

In order to compute the Lie bracket in the general case we will use the following formula from [8, p. 49].

$$[\mathbf{A}_i, \mathbf{A}_j] = \frac{d^2}{ds\,dt} q_j(s) q_i(t) q_j(s)^{-1} \bigg|_{t=s=0}.$$

We get the following results (again $\varepsilon_{jkm}$ is the Levi-Civita symbol).

$$\begin{array}{ll} [\mathbf{p}_j, \mathbf{p}_k] = [\mathbf{p}_j, \mathbf{H}_0] = [\mathbf{J}_j, \mathbf{H}_0] = 0 & \\ [\mathbf{N}_j, \mathbf{p}_k] = -\delta_{jk} \mathbf{H}_0 & [\mathbf{N}_j, \mathbf{H}_0] = -\mathbf{p}_j \\ [\mathbf{J}_j, \mathbf{p}_k] = -\sum_m \varepsilon_{jkm} \mathbf{p}_m & [\mathbf{J}_j, \mathbf{J}_k] = -\sum_m \varepsilon_{jkm} \mathbf{J}_m \\ [\mathbf{N}_j, \mathbf{N}_k] = \sum_m \varepsilon_{jkm} \mathbf{J}_m & [\mathbf{J}_j, \mathbf{N}_k] = -\sum_m \varepsilon_{jkm} \mathbf{N}_m. \end{array} \tag{2}$$

Computing these is trivial. We give the description of how in the appendix, see section A.4. Note that we are in the setting of Thm. 2.36, so the Lie algebra is the span of these generators.

**Theorem 3.12.** *The Lie algebra of the covering group $\tilde{\mathcal{P}}_+^\uparrow$ has trivial second cohomology group.*

*Proof.* Let $\theta$ be any bilinear map satisfying the defining equations. We need to define a linear map $\vartheta$ s.t. $\theta(\mathbf{A}, \mathbf{B}) = \vartheta([\mathbf{A}, \mathbf{B}])$. Since the generators form a spanning set by Thm. 2.36, we only need to define $\vartheta$ on generators. Hence for any generator $\mathbf{A}_j$ we need to define $\vartheta(\mathbf{A}_j)$ such that, if $[\mathbf{A}_k, \mathbf{A}_i] = \mathbf{A}_j$, then $\vartheta(\mathbf{A}_j) = \theta(\mathbf{A}_k, \mathbf{A}_i)$. It is clear by inspection that any basis element $\mathbf{A}_j$ is of the form $\mathbf{A}_j = \pm[\mathbf{A}_k, \mathbf{A}_i]$ for some other basis elements $\mathbf{A}_k, \mathbf{A}_i$. But $\mathbf{A}_k, \mathbf{A}_i$ might not be unique. For instance

$$[\mathbf{J}_1, \mathbf{J}_2] = [\mathbf{N}_2, \mathbf{N}_1].$$

Hence to prove that we may well-define $\vartheta$ in this manner, we need to prove that, if $[\mathbf{A}_k, \mathbf{A}_i] = \pm[\mathbf{A}'_k, \mathbf{A}'_i]$, then $\theta(\mathbf{A}_k, \mathbf{A}_i) = \pm\theta(\mathbf{A}'_k, \mathbf{A}'_i)$. By symmetry and the anticommutativity of $[-,-]$ this reduces to proving the equations

$$\begin{aligned} -\theta(\mathbf{N}_3, \mathbf{H}_0) &= -\theta(\mathbf{J}_1, \mathbf{p}_2) = \theta(\mathbf{J}_2, \mathbf{p}_1) \\ \theta(\mathbf{N}_1, \mathbf{p}_1) &= \theta(\mathbf{N}_2, \mathbf{p}_2) \\ \theta(\mathbf{J}_1, \mathbf{N}_2) &= -\theta(\mathbf{J}_2 \mathbf{N}_1) \\ -\theta(\mathbf{J}_1, \mathbf{J}_2) &= \theta(\mathbf{N}_1, \mathbf{N}_2). \end{aligned}$$

Proving these proves that $\vartheta$ is well-defined and that $\theta(\mathbf{A}, \mathbf{B}) = \vartheta([\mathbf{A}, \mathbf{B}])$ for all $\mathbf{A}, \mathbf{B}$ with $[\mathbf{A}, \mathbf{B}] \neq 0$. It remains to check that, if $[\mathbf{A}, \mathbf{B}] = 0$, then $\theta(\mathbf{A}, \mathbf{B}) = 0$. Again by symmetry and anticommutativity this reduces to the equations

$$\theta(\mathbf{p}_1, \mathbf{p}_2) = \theta(\mathbf{p}_1, \mathbf{H}_0) = \theta(\mathbf{J}_1, \mathbf{H}_0) = \theta(\mathbf{N}_1, \mathbf{p}_2) = 0.$$

As an example we prove that $\theta(\mathbf{J}_1, \mathbf{N}_2) = -\theta(\mathbf{J}_2, \mathbf{N}_1)$.

Since $\mathbf{N}_2 = -[\mathbf{J}_3, \mathbf{N}_1]$ we have the following

$$\begin{aligned} \theta(\mathbf{J}_1, \mathbf{N}_2) &= \theta(\mathbf{J}_1, -[\mathbf{J}_3, \mathbf{N}_1]) \\ &= \theta([\mathbf{J}_3, \mathbf{N}_1], \mathbf{J}_1) \\ &= -\theta([\mathbf{J}_1, \mathbf{J}_3], \mathbf{N}_1) - \theta([\mathbf{N}_1, \mathbf{J}_1], \mathbf{J}_3) && \text{(Jacobi)} \\ &= -\theta(\mathbf{J}_2, \mathbf{N}_1). \end{aligned}$$

The rest follow by a similar argument. ☐

**Corollary 3.13.** We have a bijective correspondence

$$\left\{ \begin{matrix} \rho : \mathcal{P}^\uparrow \to \mathrm{U}(\mathfrak{H}) \\ \text{projective unitary} \\ \text{representation of } \mathcal{P}^\uparrow \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \pi : \tilde{\mathcal{P}}^\uparrow \to \mathrm{U}(\mathfrak{H}) \text{ unitary} \\ \text{representation of } \tilde{\mathcal{P}}^\uparrow \\ \text{such that } \pi(0, -I_4) \in \ker p \end{matrix} \right\}$$

*Proof.* Combine the results of this section noting that the kernel of the covering map is $\{(0, \pm I_4)\}$. ☐

### 3.4 Detour - $\gamma$-matrices

We define the $\gamma$-matrices $\gamma = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$. (We have a slight abuse of notation as $\gamma^\mu$ are matrices.) We will not need these until the end of section 5, but we nonetheless define them now, as we will use the language of this section for the construction.

Define, similarly to $\phi$, the bijective linear map $\phi' : \mathbb{R}^4 \to H(2)$ by

$$\phi'(x) = x^0\sigma_0 - x^1\sigma_1 - x^2\sigma_2 - x^3\sigma_3.$$

Define now $\gamma$ by

$$\gamma(x) = \begin{bmatrix} 0 & \phi(x) \\ \phi'(x) & 0 \end{bmatrix}.$$

And write $\langle\gamma, x\rangle = \gamma(x)$. This is (sort of) meaningful once one defines $\gamma^\mu = \gamma(e_\mu)$ i.e.

$$\gamma^0 = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}, \qquad \gamma^k = \begin{bmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{bmatrix}.$$

It is an easy computation to see that (see [8, p. 72])

$$L_A\langle\gamma, x\rangle L_A^{-1} = \langle\gamma, \Lambda_A x\rangle, \qquad L_P\langle\gamma, x\rangle L_P^{-1} = \langle\gamma, Px\rangle.$$

# 4 Induced Representations

In this section we describe a class of representations, the induced representations. We explain the construction in general, and do the specific case of the Poincaré group. This section is based on [8, p. 82-92] and [6, p. 56-60].

It is a fact that every irreducible unitary representation of $\tilde{\mathcal{P}}^\uparrow$ arises as an induced representation. This motivates why we consider the induced representations. We will not prove this result, as a proof of this is beyond the scope of this thesis. See the presentation in [1] for a full proof.

## 4.1 Construction

We explain the construction of the induced representations for a general Lie group. This section is based on [8, p. 82-83].

Let $\mathcal{G}$ be a separable Lie group, (we say that a Lie group is separable if it has countably many connected components,) and $\mathcal{K} \leq \mathcal{G}$ a closed subgroup. Suppose $\tau : \mathcal{K} \to U(\mathfrak{X})$ is a unitary representation of $\mathcal{K}$ in some separable Hilbert space $\mathfrak{X}$.

The set $\mathcal{G}/\mathcal{K} = \{g\mathcal{K} | g \in \mathcal{G}\} = \{[g] | g \in \mathcal{G}\}$ is not in general a group, but $\mathcal{G}$ still acts on $\mathcal{G}/\mathcal{K}$ by $h.[g] = h.g\mathcal{K} = (hg)\mathcal{K} = [hg]$. Suppose we have a measure $\nu$ on $\mathcal{G}/\mathcal{K}$ invariant under this action, i.e.

$$\int_{\mathcal{G}/\mathcal{K}} f([g])\, d\nu([g]) = \int_{\mathcal{G}/\mathcal{K}} f([hg])\, d\nu([g])$$

for all (measurable) functions $f : \mathcal{G}/\mathcal{K} \to \mathbb{R}$ and all $h \in \mathcal{G}$.

Consider (measurable) functions $\phi : \mathcal{G} \to \mathfrak{X}$ satisfying

$$\phi(gk^{-1}) = \tau(k)\phi(g), \tag{3}$$

for all $g \in \mathcal{G}$ and $k \in \mathcal{K}$. We claim that $\|\phi(g)\|_{\mathfrak{X}}$ only depends on $[g]$:

Suppose that $[h] = [g]$ i.e. $h = gk^{-1}$. Then

$$\begin{aligned} \|\phi(h)\|_{\mathfrak{X}}^2 &= \big\langle \phi(h), \phi(h) \big\rangle_{\mathfrak{X}} \\ &= \big\langle \tau(k)\phi(g), \tau(k)\phi(g) \big\rangle_{\mathfrak{X}} \\ &= \big\langle \tau(k)^*\tau(k)\phi(g), \phi(g) \big\rangle_{\mathfrak{X}} \\ &= \|\phi(g)\|_{\mathfrak{X}}^2 \end{aligned}$$

since $\tau$ is unitary. We conclude that $\|\phi(g)\|_{\mathfrak{X}}$ only depend on $[g]$. Define

$$\mathfrak{M} = \left\{ \phi : \mathcal{G} \to \mathfrak{X} \,\middle|\, \begin{array}{c} \phi(gk^{-1}) = \tau(k)\phi(g) \ \forall g \in \mathcal{G} \forall k \in \mathcal{K} \\ \int_{\mathcal{G}/\mathcal{K}} \|\phi(g)\|_{\mathfrak{X}} \, d\nu([g]) < \infty \end{array} \right\} / \sim$$

where the relation $\sim$ is that of $\nu$-a.e. equality. The calculations above show that this is well-defined.

We claim that this is a Hilbert space with inner product

$$\langle \phi_1, \phi_2 \rangle_{\mathfrak{M}} = \int_{\mathcal{G}/\mathcal{K}} \langle \phi_1(g), \phi_2(g) \rangle_{\mathfrak{X}} \, d\nu([g]).$$

To show that this is well-defined is a similar computation as to show that $\|\phi(g)\|_{\mathfrak{X}}$ only depend on $[g]$. And the fact that this defines an inner product follows from the fact that $\langle -, - \rangle_{\mathfrak{X}}$ is an inner product. We still need to show that $\mathfrak{M}$ is complete. This we will not prove, as it will follows from Prop. 4.2.

Define the operator $\pi^\tau(h)$ by $(\pi^\tau(h)\phi)(g) = \phi(h^{-1}g)$. Then $\pi^\tau(h)\phi \in \mathfrak{M}$. We claim that the map $h \mapsto \pi^\tau(h)$ is a unitary representation of $\mathcal{G}$ in $\mathfrak{M}$.

Continuity follows from multiplication in $\mathcal{G}$ being continuous. The fact that $\pi^\tau$ is a group homomorphism is clear. To prove that $\pi^\tau$ is unitary we have the following

$$\begin{aligned} \langle \pi^\tau(h)\phi, \phi' \rangle_{\mathfrak{M}} &= \int_{\mathcal{G}/\mathcal{K}} \left\langle \phi(h^{-1}g), \phi'(g) \right\rangle_{\mathfrak{X}} d\nu([g]) \\ &= \int_{\mathcal{G}/\mathcal{K}} \langle \phi(g), \phi'(hg) \rangle_{\mathfrak{X}} \, d\nu([g]) \\ &= \langle \phi, \pi^\tau(h^{-1})\phi' \rangle_{\mathfrak{M}}. \end{aligned}$$

Hence $(\pi^\tau(h))^* = \pi^\tau(h^{-1}) = (\pi^\tau(h))^{-1}$ so $\pi^\tau$ is unitary.

**Definition 4.1.** The map $\pi^\tau : \mathcal{G} \to U(\mathfrak{M})$ defined above is called the representation of $\mathcal{G}$ induced by $\tau$ or the induced representation. $\tau$ is called the inducing representation.

## 4.2 Wigner States

Now, the description of $\mathfrak{M}$ seems rather long. Since the functions $\phi \in \mathfrak{M}$ are mostly controlled by their values on $\mathcal{G}/\mathcal{K}$ it would be sensible to think that we may view $\mathfrak{M}$ as some Hilbert space of functions on $\mathcal{G}/\mathcal{K}$. This motivates the following.

Suppose we have a canonical way of finding a representative of the cosets, i.e. a function $s : \mathcal{G}/\mathcal{K} \to \mathcal{G}$ s.t. $s(g\mathcal{K}) \in g\mathcal{K}$. Define $\mathcal{S} = s(\mathcal{G}/\mathcal{K})$ to be the image. Then $\mathcal{G}$ acts on $\mathcal{S}$ by $h.s(g\mathcal{K}) = s(hg\mathcal{K})$. This action makes the diagram

$$\begin{tikzcd} \mathcal{G}/\mathcal{K} \arrow[r, "\sim", "s"'] \arrow[d, "h."] & \mathcal{S} \arrow[d, "h."] \\ \mathcal{G}/\mathcal{K} \arrow[r, "\sim", "s"'] & \mathcal{S} \end{tikzcd}$$

commute. Hence an invariant measure $\nu$ on $\mathcal{G}/\mathcal{K}$ induces an invariant measure $\tilde{\nu}$ on $\mathcal{S}$.

Given some function $\zeta$ on $\mathcal{S}$, we want to define some function $\zeta^e$ on $\mathcal{G}$ satisfying Eqn. (3) and $\zeta^e|\mathcal{S} = \zeta$.

Let $g \in \mathcal{G}$ and $s = s(g\mathcal{K})$. Then $g \in s\mathcal{K} = g\mathcal{K}$ so there exists some unique $k \in \mathcal{K}$ such that $g = sk^{-1}$. Then

$$\zeta^e(g) = \zeta^e(sk^{-1}) = \tau(k)\zeta^e(s) = \tau(k)\zeta(s).$$

Hence this property determines $\zeta^e$ uniquely. So the map $\zeta \mapsto \zeta^e$ is well-defined.

**Proposition 4.2.** The maps

$$\begin{array}{rcl} \mathfrak{M} & \rightleftarrows & L^2(\mathcal{S}, \tilde{\nu}, \mathfrak{X}) \\ \phi & \mapsto & \phi|_{\mathcal{S}} = \zeta \\ \phi = \zeta^e & \leftarrow\!\shortmid & \zeta \end{array}$$

are mutually inverses and unitary.

*Proof.* The fact, that they preserve the inner products, follows from the fact, that either integral is over the same set, (with the above identification,) and that

$$\left\langle \zeta(s), \zeta'(s) \right\rangle_{L^2(\mathcal{S},\tilde{\nu},\mathfrak{X})} = \left\langle \phi(g), \phi'(g) \right\rangle_{\mathfrak{M}}$$

by a similar computation as in the previous section. It follows that both maps preserve the finiteness of the integrals, i.e. norms, as well.

From the calculation done above, read either left to right or right to left, it follows that these maps are inverses.

Linearity is clear by construction of either of the maps. Unitarity follows once we show that

$$\langle \phi, \zeta^e \rangle_{\mathfrak{M}} = \langle \phi|_{\mathcal{S}}, \zeta \rangle_{L^2(\mathcal{S},\tilde{\nu},\mathfrak{X})} \, .$$

But this is clear using the above. Just apply the map $\phi \mapsto \phi|_{\mathcal{S}}$ on the left side. We conclude that they are unitary. ☐

Elements of $L^2(\mathcal{S}, \nu, \mathfrak{X})$ are called Wigner states, see [8, p. 89].

These unitary maps give a unitary equivalence between $\pi^\tau$ and a unitary representation $\pi^\tau_w$ of $\mathcal{G}$ in $L^2(\mathcal{S}, \tilde{\nu}, \mathfrak{X})$. We now compute $\pi^\tau_w(g)$ explicitly.

We have $(\pi^\tau_w(h)\zeta)(s) = \pi(h)\phi(s) = \phi(h^{-1}s)$. Now $h^{-1}s \in h^{-1}s\mathcal{K}$. This set has canonical representative $h^{-1}.s$ so $h^{-1}s = (h^{-1}.s)k^{-1}$ for some (unique) $k \in \mathcal{K}$. This $k$ is given by $k = s^{-1}h(h^{-1}.s)$ and we conclude that

$$\begin{aligned}
(\pi^\tau_w(h)\zeta)(s) &= \phi(h^{-1}s) \\
&= \phi((h^{-1}.s)k^{-1}) \\
&= \tau(k)\phi(h^{-1}.s) \\
&= \tau(s^{-1}h(h^{-1}.s))\zeta(h^{-1}.s).
\end{aligned}$$

So this is the action of $\pi^\tau_w$.

## 4.3 The Dual Group

For the subgroup $\mathcal{K}$ in the construction of the induced representations, we will pick the stabilizer of $\tilde{\mathcal{P}}^\uparrow$ acting on the dual group of $\mathbb{R}^4$. To do this we first define the dual group in general.

**Definition 4.3.** Let $\mathcal{G}$ be an abelian group.

A character of $\mathcal{G}$ is a one-dimensional unitary representation (i.e. in $\mathbb{C}$) of $\mathcal{G}$. Pointwise multiplication and pointwise inversion give the set of characters the structure of a group. We call this group the dual group of $\mathcal{G}$ and denote it by

$$\widehat{\mathcal{G}} = \{\chi : \chi \,\text{character of}\, \mathcal{G}\}.$$

**Proposition 4.4.** The dual group of $\widehat{\mathbb{R}^4}$ is

$$\widehat{\mathbb{R}^4} = \left\{\chi_p = e^{i\langle p,-\rangle} : p \in \mathbb{R}^4\right\} \cong \mathbb{R}^4$$

with the isomorphism $\chi_p \mapsto p$.

*Proof.* Any one-dimensional unitary representation of $\mathbb{R}^4$ is of the form $\chi : x \mapsto e^{iy^T x}$ for some $y \in \mathbb{R}^4$. This follows by linearity and unitarity. By picking a basis according to $\langle -, - \rangle$ we are done. ☐

## 4.4 Action of $\tilde{\mathcal{P}}^\uparrow_+$ on $\widehat{\mathbb{R}^4}$

We find the stabilizer of the action and describe how the set $\mathcal{S}$ is in a natural bijection with the orbits. This section is based on [6, p. 56-60] and [8, p. 85-90].

We will identify $\widehat{\mathbb{R}^4}$ with $\mathbb{R}^4$ as in the proposition above. Note that the interpretation of the elements $p \sim \chi_p \in \widehat{\mathbb{R}^4}$ now changes. Instead of being the position they are the momentum, i.e. $p^0$ is the energy and $\bar{p}$ is the momentum in the spatial coordinates. The equations we get from this are then written in momentum-space.

We start with the general construction.

**Definition 4.5.** Suppose the group $\mathcal{G}$ acts on some set $M$. let $q \in M$. We will denote the orbits of $M$ under this action by $\mathcal{O}_q = \{g.q \in M | g \in \mathcal{G}\}$.

The stabilizer $\mathcal{G}_q = \{g \in \mathcal{G} | g.q = q\}$ will sometimes be called the isotropy group.

Note that we have a natural bijection between the sets $\mathcal{G}/\mathcal{G}_q$ and $\mathcal{O}_q$.

$$\mathcal{O}_q \ni g.q \mapsto g\mathcal{G}_q \in \mathcal{G}/\mathcal{G}_q.$$

Of course we need to prove that this is well-defined, but if $g, g'$ has $g.q = g'.q$, then they differ by an element of $\mathcal{G}_q$. This bijection is natural in the sense that it commutes with the action by $\mathcal{G}$,

$$\begin{tikzcd}
\mathcal{O}_q & \ni & g.q \arrow[r, mapsto, "h."] \arrow[d, mapsto] & (hg).q \arrow[d, mapsto] & \in & \mathcal{O}_q \\
\mathcal{G}/\mathcal{G}_q & \ni & g\mathcal{G}_q \arrow[r, mapsto, "h."] & hg\mathcal{G}_q & \in & \mathcal{G}/\mathcal{G}_q
\end{tikzcd}$$

Hence we may construct the set $\mathcal{S}$ as above by using this bijection. Then

$$\mathcal{O}_q \ni g.q \mapsto s(g\mathcal{G}_q) \in \mathcal{S}$$

is a bijection, which is natural in the above sense.

We now specialize this construction to $\mathcal{G} = \tilde{\mathcal{P}}^\uparrow$ acting on $M = \mathbb{R}^4 \trianglelefteq \mathcal{P}^\uparrow$. This action is by conjugation,

$$(b, L).(a, I_4) = (b, \Lambda_L)(a, I_4)(b, \Lambda_L)^{-1} = (b, \Lambda_L)(a, I_4)(-\Lambda_L^{-1} b, \Lambda_L^{-1}) = (\Lambda_L a, I_4).$$

Hence the orbits are

$$\mathcal{O}_q = \{\Lambda_L q | (b, L) \in \tilde{\mathcal{P}}^\uparrow\} = \{\Lambda q | \Lambda \in \tilde{\mathcal{L}}^\uparrow\},$$

and the isotropy group is

$$\tilde{\mathcal{P}}_q^\uparrow = \mathbb{R}^4 \rtimes \tilde{\mathcal{L}}_q^\uparrow,$$

where $\tilde{\mathcal{L}}_q^\uparrow = \{L \in \tilde{\mathcal{L}}^\uparrow | \Lambda_L q = q\}$.

**Definition 4.6.** We say that $\tilde{\mathcal{L}}_q^\uparrow$ is the little group of $q$.

**Proposition 4.7.** We have the following exhausting list of orbits of $\mathbb{R}^4$ under the action by $\tilde{\mathcal{P}}^\uparrow$, with $\mu > 0$

$$\begin{array}{rll}
\text{(i)} & q = (\mu, 0, 0, 0) & \mathcal{O}_q = \{p \in \mathbb{R}^4 | \langle p, p \rangle = \mu^2, p^0 > 0\}, \\
\text{(ii)} & q = (-\mu, 0, 0, 0) & \mathcal{O}_q = \{p \in \mathbb{R}^4 | \langle p, p \rangle = \mu^2, p^0 < 0\}, \\
\text{(iii)} & q = (0, \mu, 0, 0) & \mathcal{O}_q = \{p \in \mathbb{R}^4 | \langle p, p \rangle = -\mu^2\}, \\
\text{(iv)} & q = (1, 1, 0, 0) & \mathcal{O}_q = \{p \in \mathbb{R}^4 | \langle p, p \rangle = 0, p^0 > 0\}, \\
\text{(v)} & q = (-1, 1, 0, 0) & \mathcal{O}_q = \{p \in \mathbb{R}^4 | \langle p, p \rangle = 0, p^0 < 0\}, \\
\text{(vi)} & q = (0, 0, 0, 0) & \mathcal{O}_q = \{0\}.
\end{array}$$

*Proof.* It is trivial to prove that these are disjoint and exhaust $\mathbb{R}^4$. It remains to be shown that these are indeed orbits, i.e. that $\Lambda q \in \mathcal{O}_q$ for any $\Lambda \in \mathcal{L}^\uparrow$, and that for any $p \in \mathcal{O}_q$ there exists $\Lambda \in \mathcal{L}^\uparrow$, such that $p = \Lambda q$.

First note that any $p = (p^0, \bar{p}) \in \mathbb{R}^4$ can, by rotations, be mapped onto the set $\{p \in \mathbb{R}^4 : p^2 = p^3 = 0\}$. (We may in fact also assume that $p^1 \geq 0$). Now the orbits (in the set $\{p \in \mathbb{R}^4 : p^2 = p^3 = 0\}$) under boosts in the $p^1$-directions are given by

$$\begin{array}{rl}
\text{(i)} & \{p : (p^0)^2 - (p^1)^2 = \mu, \quad p^0 > 0\}, \\
\text{(ii)} & \{p : (p^0)^2 - (p^1)^2 = \mu, \quad p^0 < 0\}, \\
\text{(iii)} & \{p : (p^0)^2 - (p^1)^2 = -\mu\}, \\
\text{(iv)} & \{p : p^0 = \pm p^1, \quad p^0 > 0\}, \\
\text{(v)} & \{p : p^0 = \pm p^1, \quad p^0 < 0\}, \\
\text{(vi)} & \{0\}
\end{array}$$

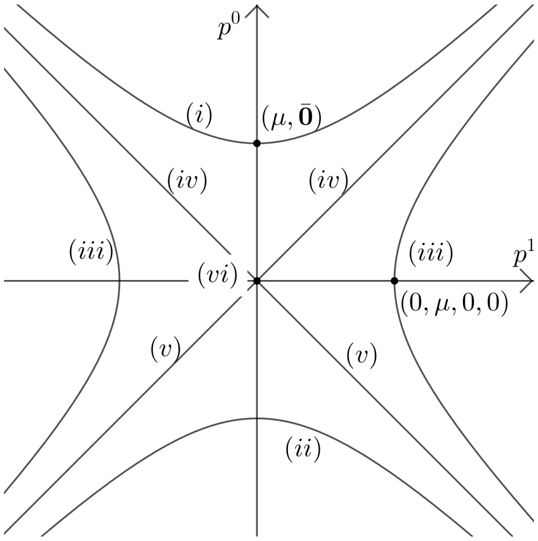

Figure 6: The orbits of $\mathbb{R}^4$ under the action by $\tilde{\mathcal{P}}^{\uparrow}$, see Prop. 4.7. Here is seen the intersection with the plane $p^2 = p^3 = 0$. This captures all the interesting parts of the orbits. Changing the parameter $\mu > 0$ we see that we exhaust the whole plane. Some of the orbits seem disconnected in this picture, namely the cases (iii), (iv) and (v). This is an artefact of only showing the intersection with some plane. In the space, $\mathbb{R}^4$, we need to rotate this picture around in the last 2 coordinates. This connects the two halves of the orbits from cases (iii), (iv) and (v).

for $\mu > 0$ a parameter. To see this note that boosts in the $p^1$-direction are given by

$$\begin{bmatrix} \cosh \omega & \sinh \omega & 0 & 0 \\ \sinh \omega & \cosh \omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

for $\omega \in \mathbb{R}$. Hence the orbits are just the disjoint hyperbolas claimed above. This proves that the six claimed orbits are indeed stable under the action. They are contained in only one orbit each. It remains to show that no two are contained in the same orbit.

On any orbit $\langle p, p \rangle$ is constant. Moreover, by definition of $\mathcal{L}^{\uparrow}$, we see that the action preserves the sign of $p^0$. Hence the list above are exactly he list of orbits under the action of $\mathcal{L}^{\uparrow}$, i.e. also that of $\tilde{\mathcal{P}}^{\uparrow}$. ☐

The interpretation here is that $\bar{\mathbf{p}}$ is the momentum, $p^0$ is the energy and $\mu$ is the mass (in cases (i) and (ii)), see [6, p. 63]. We restrict to the case of positive mass and energy, that is case (i). We may parameterize the orbit by

$$\left\{ \left( \sqrt{\mu^2 + \bar{\mathbf{p}}^2}, \bar{\mathbf{p}} \right) \middle| \bar{\mathbf{p}} \in \mathbb{R}^3 \right\}.$$

The following may as well be applied to the case (ii), we only need to change a few signs here and there. We only do this for the case (i) for clarity of the arguments. Case (ii) correspond to positive mass and negative energy. With this parametrization we induce a measure $m_3$, $dm_3(\bar{\mathbf{p}}) = d^3\bar{\mathbf{p}}$ from the Lebesgue measure on $\mathbb{R}^3$. Using this we construct the measure $\nu$ to have density $\frac{1}{\sqrt{\mu^2+\bar{\mathbf{p}}^2}} = \frac{1}{p^0}$ with respect to this induced measure, i.e.

$$d\nu = \frac{d^3\bar{\mathbf{p}}}{\sqrt{\mu^2 + \bar{\mathbf{p}}^2}} = \frac{d^3\bar{\mathbf{p}}}{p^0}.$$

**Lemma 4.8.** The measure $\nu$ defined above is invariant under the action of $\tilde{\mathcal{P}}^{\uparrow}$.

*Proof.* It is immediate that this measure is invariant under the rotations. For the boosts the transformations introduce a factor of the Jacobian. We need to prove, that this factor corresponds to, how the factor $p^0 = \sqrt{\mu^2 + \bar{\mathbf{p}}^2}$ changes. Since any map (taking $p$ to $p'$) may be seen as the product of two boosts (and possibly also a rotation, but this is not important). The first taking $p$ to $q = (\mu, 0, 0, 0)$ and the second taking $q$ to $p'$. Hence it is enough to check for boosts away from $q = (\mu, 0, 0, 0)$, i.e. when evaluating the Jacobian for this transformation, we may just evaluate it at $q$.

For a boost with velocity $\bar{\mathbf{v}}$ we have $p = \Lambda q$, so $\bar{\mathbf{p}} = \mu\gamma(\nu)\bar{\mathbf{v}}$, and $p^0 = \mu\gamma(\nu)$. Now the Jacobian is the determinant of the bottom right $3 \times 3$ matrix in the boost. This matrix, written in a basis with $\widehat{\nu} = \bar{\mathbf{v}}/\nu$ as the first vector, is

$$I_3 + \frac{\gamma - 1}{\bar{\mathbf{v}}^2}\bar{\mathbf{v}}\bar{\mathbf{v}}^T = I_3 + \begin{pmatrix} \nu \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \nu & 0 & 0 \end{pmatrix} \frac{\gamma - 1}{\nu^2} = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence the determinant is $\gamma = \frac{p^0}{\mu}$. So

$$d\nu(p) = d\nu(\Lambda q) = \frac{d^3\overline{\Lambda q}}{(\Lambda q)^0} = \frac{p^0}{\mu}\frac{d^3\bar{\mathbf{q}}}{p^0} = \frac{d^3\bar{\mathbf{q}}}{q^0} = d\nu(q)$$

where we have written $q = (\mu, 0, 0, 0)$ as $(q^0, \bar{\mathbf{q}})$ for clarity. This proves that the measure is invariant. ☐

We are now ready to explicitly state how all representations arises as induced representations.

**Theorem 4.9** ([6, p. 50] or [8, Thm. 3.7.]). *Let $\pi$ be an irreducible unitary representation of $\tilde{\mathcal{P}}^\uparrow$. Then $\pi \simeq \pi^\tau$ is equivalent to an induced representation where $\tau(a, L) = \chi_q(a)\sigma(L)$ for some unitary representation $\sigma$ of $\tilde{\mathcal{L}}_q^\uparrow$ in $\mathfrak{X}$. Furthermore, $\sigma$ is determined uniquely up to equivalence and $\mathcal{O}_q$ is determined uniquely.*

We now want to find $\tilde{\mathcal{L}}_q^\uparrow$ in the case $q = (\mu, 0, 0, 0)$.

**Proposition 4.10.** Let $q = (\mu, 0, 0, 0)$ then $\tilde{\mathcal{L}}_q^\uparrow = \{L_U, L_P L_U : U \in SU(2)\}$.

*Proof.* We have, similarly as the comment after Lemma 3.9, that $\tilde{\mathcal{L}}_q^\uparrow$ is exactly the set of $L_A$ for which $A\sigma_0 A^* = \sigma_0$ and products of such with $L_P$. Now, $\sigma_0 = I_2$ and we conclude, that $A$ is unitary. ☐

## 4.5 Irreducibility

Having now constructed the induced representations we give necessary and sufficient conditions for their irreducibility. This section is based on [8, p. 91-92].

We use the Wigner states on $\mathcal{G} = \tilde{\mathcal{P}}^\uparrow$ and $\mathcal{K} = \tilde{\mathcal{P}}_q^\uparrow = \mathbb{R}^4 \rtimes \tilde{\mathcal{L}}_q^\uparrow$. Let $\sigma : \tilde{\mathcal{L}}_q^\uparrow \to U(\mathfrak{X})$ be a unitary representation of $\tilde{\mathcal{L}}_q^\uparrow$ and let $\tau$ be given by $\tau(a, L) = \chi_q(a)\sigma(L) = e^{i\langle q, a\rangle}\sigma(L)$.

We define the canonical representatives using the correspondence between $\mathcal{G}/\mathcal{K}$ and $\mathcal{O}_q$ given by $[g] \mapsto p = g.q$. For case (i) let $s([g]) = (0, L_p)$, where $L_p$ correspond to the boost $\Lambda_{L_p}$, the unique boost $\Lambda$ with $\Lambda q = p = g.q$. There are two such $L_p$ by Lem. 3.8. Any one of them will work, but for the sake of explicitly picking one impose (just for this) a total ordering (say lexiographic) on $M_{4\times 4}$ and pick the larger according to this ordering.

This gives a representation $\pi_w^\tau$ of $\tilde{\mathcal{P}}^\uparrow$ in $L^2(\mathcal{S}, \tilde{\nu}, \mathfrak{X})$. To calculate it explicitly we have that

$$(a, L).(0, L_p) = (0, L_{\Lambda_L p})$$

since we identify $(0, L_p) \sim p \in \mathcal{O}_q$, on which $\tilde{\mathcal{P}}^\uparrow$ acts by left multiplication. For the argument $s^{-1}h(h^{-1}.s)$ we have

$$\begin{aligned}(0, L_p)^{-1}(a, L)((a, L)^{-1}.(0, L_p)) &= (0, L_p^{-1})(a, L)(0, L_{\Lambda_L^{-1}p}) \\ &= (0, L_p^{-1})(a, LL_{\Lambda_L^{-1}p}) \\ &= (\Lambda_{L_p}^{-1}a, L_p^{-1}LL_{\Lambda_L^{-1}p}).\end{aligned}$$

Moreover,

$$\left\langle q, \Lambda_{L_p}^{-1} a \right\rangle = \langle \Lambda_{L_p} q, a \rangle = \langle p, a \rangle .$$

We conclude that

$$\begin{aligned}\left(\pi_w^\tau(a, L)\zeta\right)(0, L_p) &= \tau\left((0, L_p)^{-1}(a, L)((a, L)^{-1}.(0, L_p))\right)\zeta\left(0, L_{\Lambda_L^{-1}p}\right) \\ &= e^{i\langle p, a\rangle}\sigma\left(L_p^{-1}LL_{\Lambda_L^{-1}p}\right)\zeta\left(0, L_{\Lambda_L^{-1}p}\right)\end{aligned}$$

is the explicit form of the representation.

With the construction of $\mathcal{S}$ above, we may instead think of functions on $\mathcal{O}_q$, as $\mathcal{S}$ and $\mathcal{O}_q$ are in a natural bijective correspondence c.f. the previous section. The representation thus becomes

$$(\pi_w^\tau(a, L)\zeta)(p) = e^{i\langle p, a\rangle}\sigma\left(L_p^{-1}LL_{\Lambda_L^{-1}p}\right)\zeta\left(\Lambda_L^{-1}p\right). \tag{4}$$

It is a representation of $\tilde{\mathcal{P}}^\uparrow$ in the Hilbert space $\mathfrak{M}_q = L^2(\mathcal{O}_q, \nu, \mathfrak{X})$.

We are now almost ready to prove the irreducibility of the induced representations. In order to do this we will use Schur's lemma (Thm. 2.27). Hence we need to consider (some of) the bounded linear operators.

**Lemma 4.11.** Let $B : L^2(S, \nu, \mathfrak{X}) \to L^2(S, \nu, \mathfrak{X})$ be a bounded linear operator. here $S \subset \mathbb{R}^n$ is some set, $\nu$ is some measure, which is finite on all compact sets and $\mathfrak{X}$ is some separable Hilbert space. Suppose that

$$Be^{i(-)\cdot a} = e^{i(-)\cdot a}B$$

For all $a \in \mathbb{R}^n$. Here $e^{i(-)\cdot a}$ is the multiplication operator $(e^{i(-)\cdot a}f)(p) = e^{ip\cdot a}f(p)$. Then $B$ is an operator on $\mathfrak{X}$, i.e. $(Bf)(p) = B(p)f(p)$ for all $p \in S$.

Some of the ideas for this proof was given to me by my advisor Jan Philip Solovej.

*Proof.* We first prove, that $B\hat{g} = \hat{g}B$ for all Schwartz functions $\hat{g} \in \mathcal{S}(\mathbb{R}^n)$. Here $\hat{g}$ is the operator defined by $(\hat{g}f)(p) = \hat{g}(p)f(p)$. Since the Fourier transform is a bijection $\mathcal{S} \to \mathcal{S}$, these are in fact all Schwartz functions.

Write $\hat{g} = \int_{\mathbb{R}^n} e^{-ip\cdot x}g(x)\,dx$ (with some power of $\sqrt{2\pi}$ hidden in the measure) as a Fourier transform. Then we have

$$\begin{aligned}(B\hat{g})f &= B(\hat{g}f) \\ &= B\left(\int_{\mathbb{R}^n} e^{-i(-)\cdot x}f(-)g(x)\,dx\right) \\ &\overset{*}{=} \int_{\mathbb{R}^n} Be^{-i(-)\cdot x}f(-)g(x)\,dx \\ &= \int_{\mathbb{R}^n} e^{-i(-)\cdot x}(Bf)(-)g(x)\,dx \\ &= \hat{g}Bf\end{aligned}$$

for all $f \in L^2(S, \nu, \mathfrak{X})$ for which $*$ holds. We prove it holds for all such $f$. We consider the inner products. Let $h \in L^2(S, \nu, \mathfrak{X})$ and write $f_x(p) = e^{-ip\cdot x}f(p)$. Then

$$\begin{aligned}\left\langle h, B\left(\int_{\mathbb{R}^n} f_x g(x)\,dx\right)\right\rangle_{L^2} &= \left\langle B^*h, \int_{\mathbb{R}^n} f_x g(x)\,dx\right\rangle_{L^2} \\ &= \int_S \left\langle (B^*h)(p), \int_{\mathbb{R}^n} f_x(p)g(x)\,dx\right\rangle_{\mathfrak{X}} d\nu(p) \\ &= \int_S\int_{\mathbb{R}^n} \langle (B^*h)(p), f_x(p)g(x)\rangle_{\mathfrak{X}}\,dx\,d\nu(p) \\ &\overset{**}{=} \int_{\mathbb{R}^n}\int_S \langle (B^*h)(p), f_x(p)g(x)\rangle_{\mathfrak{X}}\,d\nu(p)\,dx \\ &= \int_{\mathbb{R}^n} \langle B^*h, f_x\rangle_{L^2}\,\overline{g(x)}\,dx \\ &= \int_{\mathbb{R}^n} \langle h, Bf_x\rangle_{L^2}\overline{g(x)}\,dx \\ &= \left\langle h, \int_{\mathbb{R}^n} Bf_x g(x)\,dx\right\rangle_{L^2}.\end{aligned}$$

We used Fubini's thm. in $**$. By construction $g \in \mathcal{S} \subset L^1$ and $\|f_x(p)\|_{\mathfrak{X}} = \|f(p)\|_{\mathfrak{X}}$. Hence we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} \int_S \left|\langle (B^*h)(p), f_x(p)g(x)\rangle_{\mathfrak{X}}\right| \, d\nu(p)\, dx \\
&\leq \int_{\mathbb{R}^n} \int_S \|(B^*h)(p)\|_{\mathfrak{X}} \, \|f_x(p)\|_{\mathfrak{X}} \, |g(x)| \, d\nu(p)\, dx \\
&\leq \int_{\mathbb{R}^n} |g(x)|\, dx \left( \int_S \|(B^*h)(p)\|_{\mathfrak{X}}^2 \, d\nu(p) \right)^{1/2} \left( \int_S \|f(p)\|_{\mathfrak{X}}^2 \, d\nu(p) \right)^{1/2} \\
&= \|g\|_{L^1} \, \|B^*h\|_{L^2(S,\nu,\mathfrak{X})} \, \|f\|_{L^2(S,\nu,\mathfrak{X})} < \infty.
\end{aligned}$$

So we may conclude as above.

Since $\mathfrak{X}$ is separable this proves that $*$ holds for all $f \in L^2(S, \nu, \mathfrak{X})$, i.e. that $B$ commutes with multiplication by ($\mathbb{C}$-valued) Schwartz-functions.

We now want to prove that $1_\Omega Bf = B 1_\Omega f$ for all $f \in L^2(S, \nu, \mathfrak{X})$. This can be done by approximating $1_\Omega$ by Schwartz-functions using an approximate unit $(u_k)_{k\in\mathbb{N}}$. Then $(u_k * 1_\Omega) \in \mathcal{S}(\mathbb{R}^n)$ converges to $1_\Omega$ for $k \to \infty$. It remains to be shown that $((u_k) * 1_\Omega)f \to 1_\Omega f$ for all $f \in L^2(S, \nu, \mathfrak{X})$. This involves taking the 2-norm of the difference and using dominated convergence. We leave out the details. Now, the previous gives, $B(u_k * 1_\Omega)f = (u_k * 1_\Omega)Bf$. Taking limits and using the continuity of $B$ we get the desired.

Now define for all $p \in S$ the operator $B(p) : \mathfrak{X} \to \mathfrak{X}$ by

$$B(p)\xi = B(\xi 1_\Omega)(p)$$

for some bounded set $\Omega \subset S$ with $p \in \Omega$. To prove that this is well-defined consider the following for any $h \in L^2(S, \nu, \mathfrak{X})$

$$\begin{aligned}
B(h)(p) &= B(h1_\Omega)(p) + B(h(1 - 1_\Omega))(p) \\
&= B(h1_\Omega)(p) + 1_\Omega(p)B(h(1 - 1_\Omega))(p) \\
&= B(h1_\Omega)(p) + B(1_\Omega h(1 - 1_\Omega))(p) \\
&= B(h1_\Omega)(p).
\end{aligned}$$

Using this on $1_{\Omega'}$ and $1_\Omega$ for two sets $\Omega', \Omega$ s.t. $p \in \Omega \cap \Omega'$ we see that

$$B(\xi 1_\Omega)(p) = 1_{\Omega'}(p)B(\xi 1_\Omega)(p) = B(\xi 1_\Omega 1_{\Omega'})(p) = 1_\Omega(p)B(\xi 1_{\Omega'})(p) = B(\xi 1_{\Omega'})(p).$$

Hence $B(p)$ is well-defined.

Now to prove that $(Bf)(p) = B(p)f(p)$ suppose first that $f$ is simple (i.e. taking only finitely many values) and compactly supported. Then $f = \sum_{i=1}^N \xi_i 1_{\Omega_i}$ and so

$$\left(B \textstyle\sum_{i=1}^N \xi_i 1_{\Omega_i}\right)(p) = \textstyle\sum_{i=1}^N 1_{\Omega_i}(p)B(p)\xi_i = B(p)f(p).$$

Now the set of compactly supported simple functions is dense. To see this, note that any function $f$ can be written as the limit of simple functions $f^j \to f$, and every $f^j$ can be written as a limit of compactly supported simple functions $f^{(j,k)} \to f^j$. Then the net $(f^{(j,k)})_{(j,k)}$ (ordered by $(j,k) \geq (j', k')$ if $j \geq j'$ and $k \geq k'$) is seen to converge to $f$.

In conclusion we may extend by continuity to see that $(Bf)(p) = B(p)f(p)$ for all $f \in L^2$. ☐

Finally we may prove the main theorem, that the induced representations constructed above are irreducible.

**Theorem 4.12.** *Let $\tau(a, L) = \chi_q(a)\sigma(L)$ be defined as above, with some unitary representation $\sigma$ of $\tilde{\mathcal{L}}_q^\uparrow$ in $\mathfrak{X}$. Then the following are equivalent*

- *$\sigma$ is irreducible.*
- *$\pi_w^\tau$ is irreducible.*

*Proof.* Suppose first that $\sigma$ is reducible. Then there exists some proper invariant subspace $\mathfrak{X}'$ for $\sigma$. We immediately get, that the space $L^2(\mathcal{O}_q, \nu, \mathfrak{X}')$ is a proper invariant subspace for $\pi_w^\tau$. We conclude that $\pi_w^\tau$ is reducible.

Suppose now that $\sigma$ is irreducible.

We use Schur's lemma (Thm. 2.27) to prove that $\pi_w^\tau$ is irreducible. Hence let $B : \mathfrak{M}_q \to \mathfrak{M}_q$ be bounded linear with $\pi_w^\tau(a, L)B = B\pi_w^\tau(a, L)$ for all $(a, L) \in \tilde{\mathcal{P}}^\uparrow$.

First, using this for elements $(a, I_4)$ we see that

$$\left(B(e^{i\langle -,a\rangle}\zeta)\right)(p) = (B\pi_w^\tau(a, I_4)\zeta)(p) = (\pi_w^\tau(a, I_4)B\zeta)(p) = e^{i\langle p,a\rangle}(B\zeta)(p).$$

Hence by Lemma 4.11 we have that $B$ is action by an operator on $\mathfrak{X}$ (dependent on $p$), i.e. $(B\zeta)(p) = B(p)\zeta(p)$.

Extend the functions $B\zeta$ and $\zeta$ by the map $\zeta \mapsto \zeta^e$ of Prop. 4.2. Hence they satisfy Eqn. (3). Extend $B$ to be constant on the cosets $(0, L_p)\tilde{\mathcal{L}}_q^\uparrow$.

We may thus still think of $B$ as an (operator-valued) function on $\mathcal{O}_q$. Combining this we get for $L \in \tilde{\mathcal{L}}^\uparrow$ that

$$(B\zeta)\left((0, L_p)(0, L)^{-1}\right) = B(p)\zeta\left((0, L_p)(0, L)^{-1}\right).$$

Using Eqn. (3) on both sides we arrive at

$$\big(\sigma(L)B(p)\big)\zeta(p) = \sigma(L)(B\zeta)(p) = (B(p)\sigma(L))\,\zeta(p).$$

Since this holds for all functions $\zeta$, we conclude (by picking enough such $\zeta$ for each $p$, say a family having as values a basis for $\mathfrak{X}$) that

$$\sigma(L)B(p) = B(p)\sigma(L)$$

for all $L \in \tilde{\mathcal{L}}_q^\uparrow$. Since $\sigma$ is irreducible, we get by Schur's Lemma (Thm. 2.27), that $B(p) = b(p)\,\mathrm{id}_{\mathfrak{X}}$ is multiplication by a function of $p$.

Now, using the assumption on elements of the form $(0, L)$ we get

$$\begin{aligned} b\left(\Lambda_L^{-1}\right)\sigma\left(L_p^{-1}LL_{\Lambda_L^{-1}p}\right)\zeta\left(\Lambda_L^{-1}p\right) &= \sigma\left(L_p^{-1}LL_{\Lambda_L^{-1}p}\right)b\left(\Lambda_L^{-1}\right)\zeta\left(\Lambda_L^{-1}p\right) \\ &= (\pi_w^\tau(0, L)B\zeta)(p) \\ &= (B\pi_w^\tau(0, L)\zeta)(p) \\ &= b(p)\sigma\left(L_p^{-1}LL_{\Lambda_L^{-1}p}\right)\zeta\left(\Lambda_L^{-1}p\right). \end{aligned}$$

for all $\zeta$. This is

$$\big(b(\Lambda_L^{-1}p) - b(p)\big)(\pi_w^\tau(0, L)\zeta)(p) = 0.$$

Picking enough $\zeta$ for each $L$ we conclude that $b(\Lambda_L^{-1}p) = b(p)$ for all $L \in \tilde{\mathcal{L}}^\uparrow$. So $b$ is constant on the orbit $\mathcal{O}_q$. We conclude that $B = b\,\mathrm{id}_{\mathfrak{H}}$ is multiplication by a constant. Another application of Schur's Lemma (now in the opposite direction) proves that $\pi_w^\tau$ is irreducible. ☐

## 4.6 Concluding Remarks

We have by Thm. 4.9, that all the irreducible unitary representations are given as induced ones. On the other hand Thm. 4.12 gives an exact criterion, for when an induced representation is irreducible. Hence this gives a full classification of all the irreducible unitary representations of the covering group of the Poincaré group.

In order to give a more explicit description of all the irreducible unitary representations of $\tilde{\mathcal{P}}^\uparrow$, we need to find all the irreducible unitary representations of $SU(2)$, and see which of these extend to representations of $\tilde{\mathcal{L}}_q^\uparrow$. See [6, p. 68] for a description of these.

The important thing to note, is that these unitary representations are in bijective correspondence with the non-zero integers $r = 0, 1, 2, \ldots$. These correspond to the spin of the elementary particles, in the sense that the spin is the given by $\frac{r}{2}$.

Together with Thm. 4.9 we see, that the irreducible unitary representations are parameterized by their mass $\mu$ and spin $\frac{r}{2}$. For the irreducible unitary representations of $\tilde{\mathcal{P}}_+^\uparrow$ we do not need to check for such extensions, as the little group here is $SU(2)$, see [6, p. 60].

# 5 The Dirac Equation

As the final part of this thesis we consider an example of an induced representation, namely that of positive mass and spin-1/2. This will lead to the Dirac equation in covariant form. This section is based on [8, p. 93-97].

## 5.1 The Inducing Representation

The case of positive mass and spin-1/2 correspond to the inducing representation $\tau$ with $\sigma(L) = L$ in $\mathbb{C}^4$ [8, p. 93]. The representation $\sigma$ is not irreducible. All $L \in \tilde{\mathcal{L}}_q^{\uparrow}$ have the form $L = \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}$, with $U \in SU(2)$, or they are a product of such with $L_P$. Hence the subspaces $\mathfrak{X}^+ = \{(\nu, \nu) : \nu \in \mathbb{C}^2\} = \frac{I_4 + L_p}{2}\mathbb{C}^4$ and $\mathfrak{X}^- = \{(\nu, -\nu) : \nu \in \mathbb{C}^2\} = \frac{I_4 - L_P}{2}\mathbb{C}^4$ are invariant. By inspection $\mathfrak{X}^{\pm}$ have no proper invariant subspaces, so $\sigma$ is irreducible (using the definition) when seen as a representation in $\mathfrak{X}^{\pm} = Q^{\pm}\mathbb{C}^4$ where $Q^{\pm} = \frac{I_4 \pm L_P}{2}$.

Note that $\sigma$ is a unitary representation. Thus the induced representation $\pi^{\tau}$ is an irreducible unitary representation of $\tilde{\mathcal{P}}^{\uparrow}$ in $\mathfrak{M}^{\pm} = Q^{\pm}\mathfrak{M} \cong L^2(\mathcal{O}_q, \nu, \mathfrak{X}^{\pm})$.

## 5.2 Covariant States

We do a construction akin to the construction of the Wigner States. Let $\mathcal{G}$ be a separable Lie group and $\mathcal{K} \leq \mathcal{G}$ be a closed subgroup. We are given an inducing unitary representation $\tau$ of $\mathcal{K}$

Suppose that $\tau = \tilde{\tau}|_{\mathcal{K}}$ is the restriction of a (not necessarily unitary) representation $\tau$ of the whole group. Again we consider function $\phi$ on $\mathcal{G}$ satisfying Eqn. (3). We may then construct new functions (called covariant states) $\tilde{\psi}(g) = \tilde{\tau}(g)\phi(g)$, prove that they are constant on the cosets, define an inner product of such functions, and prove, that the set $\mathfrak{C}$ of such functions is a Hilbert space isomorphic to $\mathfrak{M}$. The details may be found in [8, p. 94].

The important relation to note is that for covariant states we have $\tilde{\psi}(s) = \tilde{\tau}(s)\zeta(s)$ for the canonical representatives $s\mathcal{K} \ni s \sim p \in \mathcal{O}_q$.

## 5.3 Covariant Dirac Equation

For the Poincaré group the inducing representation $\sigma$ is the restriction of $\tilde{\sigma}(L) = L$ for all $L \in \tilde{\mathcal{P}}^{\uparrow}$. This is clearly a representation. We may thus use the above construction to get covariant states $\tilde{\psi}$. Since we may pick our canonical representatives to be on the orbit $\mathcal{O}_q$, we have $\tilde{\psi}(p) = \tilde{\tau}(0, L_p)\zeta(p) = L_p\zeta(p)$ for $\zeta \in \mathfrak{M}_q$.

Still we have that the representation $\sigma$, hence also $\pi^{\tau}$, is not irreducible. We have the projection operators $Q^{\pm}$ onto the invariant subspaces. On $\mathfrak{M}$ this is just $Q^{\pm}$ again, but how are the projection operators on $\mathfrak{C}$? To solve this consider for $\zeta \in \mathfrak{M}_q^{\pm}$

$$\tilde{\psi}(p) = L_p\zeta(p) = L_pQ^{\pm}\zeta(p) = (L_pQ^{\pm}L_p^{-1})L_p\zeta(p) = (L_pQ^{\pm}L_p^{-1})\tilde{\psi}(p).$$

Thus the operator $L_pQ^{\pm}L_p^{-1}$ is trivial on $\mathfrak{C}_q^{\pm} = L_p\mathfrak{M}_q^{\pm}$ (we have a slight abuse of notation here, the p in the index varies as the argument of the function $\tilde{\psi}$) and zero on $L_p\mathfrak{M}_q^{\mp}$. It is thus a projection.

To write out this operator in full detail we need to recall the $\gamma$-matrices from section 3.4. The important relation is that $L\langle\gamma, x\rangle L^{-1} = \langle\gamma, \Lambda_L x\rangle$. Using this on $x = q$ (a formal equality) and $L = L_p$ we get

$$\langle\gamma, p\rangle = \langle\gamma, \Lambda_{L_p} q\rangle = L_p\langle\gamma, q\rangle L_p^{-1} = L_p\gamma^0\mu L_p^{-1} = \mu L_p L_P L_p^{-1}.$$

Now, writing this operator out in full we have

$$L_pQ^{\pm}L_p^{-1} = \frac{I_4 \pm L_pL_PL_p^{-1}}{2} = \frac{\mu I_4 \pm \langle\gamma, p\rangle}{2\mu}.$$

Restricting to $\tilde{\psi} \in \mathfrak{C}_q^{\pm}$ we get $\langle \gamma, p \rangle \tilde{\psi}(p) = \pm\mu\tilde{\psi}(p)$. Considered only on the subspace $\mathfrak{C}_q^{+}$ we get the Dirac Equation in covariant form [8, p. 97]

$$\langle \gamma, p \rangle \tilde{\psi}(p) - \mu\tilde{\psi}(p) = 0.$$

This equation is written in momentum-space (i.e. p-space). To transform the equation into position-space (i.e. $x$-space) we need to use the Fourier transform. We will not do this but refer to [8, p. 97-102] for the details. One thing to note is that the operator of multiplication by $p$ transform into a differentiation operator (with some factor of $i$ as well). Thus, in position-space, the Dirac Equation is a first-order partial differential equation,

$$i \sum_{\mu} \gamma^{\mu} \frac{\partial}{\partial x^{\mu}} \psi(x) - \mu\psi(x) = 0,$$

see [8, p. 101]. This is similar to the Schrödinger equation (though this one is second order) from classical quantum mechanics, see [2].

# 6 Conclusion and Perspective

We have in section 4 described all the irreducible unitary representations of the covering group of the Poincaré group corresponding to positive mass and energy. Combining this with the results from section 3, we thus get all the projective unitary representations of the Poincaré group. Hence all the elementary particles with non-zero mass and positive energy. We have done this by lifting to representations of the covering group and through the method of induced representations.

We found that all these (projective) representations are classified by two parameters, corresponding to the mass and spin of the corresponding elementary particle.

In this presentation we have worked with space inversion and for particles with non-zero mass and positive energy. This leads to some obvious ways to pursue further work. Namely including time reversal as well, or working with other orbits, corresponding to particles with other parameters for their mass. Having a mass of zero is also a relevant physical setting, for instance photons, hence this orbit is of particular importance as well.

Another possible extension is to allow the existence of charge. We have worked in a universe without charge, but one might include charge and thus also a charge-inversion operator akin to space and time inversions resp. reversals. An important result to note here is the CPT-theorem, see for instance [7].

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# A Postponed Proofs and Arguments

Here we give the proofs of some results.

## A.1 Defining Equations for $\mathcal{L}$

The defining equations are

$$\eta_{\rho\tau} = \sum_{\mu,\nu} \Lambda^{\mu}{}_{\rho}\eta_{\mu\nu}\Lambda^{\nu}{}_{\tau}.$$

Written out we get

$$\begin{aligned}
1 &= (\Lambda^{0}{}_{0})^2 - (\Lambda^{1}{}_{0})^2 - (\Lambda^{2}{}_{0})^2 - (\Lambda^{3}{}_{0})^2 && (\rho = \tau = 0)\\
-1 &= (\Lambda^{0}{}_{1})^2 - (\Lambda^{1}{}_{1})^2 - (\Lambda^{2}{}_{1})^2 - (\Lambda^{3}{}_{1})^2 && (\rho = \tau = 1)\\
-1 &= (\Lambda^{0}{}_{2})^2 - (\Lambda^{1}{}_{2})^2 - (\Lambda^{2}{}_{2})^2 - (\Lambda^{3}{}_{2})^2 && (\rho = \tau = 2)\\
-1 &= (\Lambda^{0}{}_{3})^2 - (\Lambda^{1}{}_{3})^2 - (\Lambda^{2}{}_{3})^2 - (\Lambda^{3}{}_{3})^2 && (\rho = \tau = 3)\\
0 &= \Lambda^{0}{}_{1}\Lambda^{0}{}_{0} - \Lambda^{1}{}_{1}\Lambda^{1}{}_{0} - \Lambda^{2}{}_{1}\Lambda^{2}{}_{0} - \Lambda^{3}{}_{1}\Lambda^{3}{}_{0} && (\rho = 1, \tau = 0)\\
0 &= \Lambda^{0}{}_{2}\Lambda^{0}{}_{0} - \Lambda^{1}{}_{2}\Lambda^{1}{}_{0} - \Lambda^{2}{}_{2}\Lambda^{2}{}_{0} - \Lambda^{3}{}_{2}\Lambda^{3}{}_{0} && (\rho = 2, \tau = 0)\\
0 &= \Lambda^{0}{}_{2}\Lambda^{0}{}_{1} - \Lambda^{1}{}_{2}\Lambda^{1}{}_{1} - \Lambda^{2}{}_{2}\Lambda^{2}{}_{1} - \Lambda^{3}{}_{2}\Lambda^{3}{}_{1} && (\rho = 2, \tau = 1)\\
0 &= \Lambda^{0}{}_{3}\Lambda^{0}{}_{0} - \Lambda^{1}{}_{3}\Lambda^{1}{}_{0} - \Lambda^{2}{}_{3}\Lambda^{2}{}_{0} - \Lambda^{3}{}_{3}\Lambda^{3}{}_{0} && (\rho = 3, \tau = 0)\\
0 &= \Lambda^{0}{}_{3}\Lambda^{0}{}_{1} - \Lambda^{1}{}_{3}\Lambda^{1}{}_{1} - \Lambda^{2}{}_{3}\Lambda^{2}{}_{1} - \Lambda^{3}{}_{3}\Lambda^{3}{}_{1} && (\rho = 3, \tau = 1)\\
0 &= \Lambda^{0}{}_{3}\Lambda^{0}{}_{2} - \Lambda^{1}{}_{3}\Lambda^{1}{}_{2} - \Lambda^{2}{}_{3}\Lambda^{2}{}_{2} - \Lambda^{3}{}_{3}\Lambda^{3}{}_{2} && (\rho = 3, \tau = 2)
\end{aligned}$$

## A.2 Classification of the Lorentz Group

Here we give a proof of Prop. 2.9. We restate the result for convenience.

**Proposition A.1.** For any $\Lambda \in \mathcal{L}_+^{\uparrow}$ there exists $\bar{v} \in \mathbb{R}^3, R \in SO(3)$ such that $\Lambda = \Lambda^{v}(\bar{v})\Lambda^{R}(R)$. The assignment $\Lambda \mapsto (\bar{v}, R)$ is continuous.

*Proof.* Let $\Lambda = (\Lambda^{\mu}{}_{\nu})$ be a Lorentz transformation.

Note that then $\Lambda^T$ is a Lorentz transformation as well, since $\Lambda\eta\Lambda^T\eta\Lambda = \Lambda\eta\eta = \Lambda$, so $\Lambda\eta\Lambda^T = \eta^{-1} = \eta$.

Define $\bar{v}$ by $\bar{v}_i = \frac{\Lambda^{i}{}_{0}}{\Lambda^{0}{}_{0}}$.

Suppose first that $\bar{v} = 0$. Then $\Lambda^T = \begin{bmatrix} 1 & 0 \\ \bar{z} & R \end{bmatrix}$ for some $3 \times 3$ matrix $R$ and some vector $\bar{z} \in \mathbb{R}^3$. First, $1 = \langle e_0, e_0 \rangle = \langle \Lambda^T e_0, \Lambda^T e_0 \rangle = 1 - \bar{z}^2$. We conclude that $\bar{z} = 0$. By the defining relation we now have, that

$$x^0 y^0 - (R\bar{x})^T R\bar{y} = x^0 y^0 - \bar{x}^T \bar{y}$$

for all $x, y \in \mathbb{R}^4$. Hence $R$ preserves the inner product of $\mathbb{R}^3$ and since $\det R = \det \Lambda = 1$ we have that $R \in SO(3)$.

Suppose now that $\bar{v} \neq 0$. Then

$$\bar{v}^2 = \frac{\sum_{k=1}^{3}(\Lambda^{k}{}_{0})^2}{\Lambda^{0}{}_{0}} = \frac{(\Lambda^{0}{}_{0})^2 - 1}{(\Lambda^{0}{}_{0})^2}.$$

And

$$\gamma(\bar{v}) = \frac{1}{\sqrt{1 - \bar{v}^2}} = \Lambda^{0}{}_{0}.$$

So that

$$\frac{\gamma - 1}{\bar{v}^2} = \frac{(\Lambda^{0}{}_{0})^2}{\Lambda^{0}{}_{0} + 1}.$$

Then

$$\Lambda(-\bar{v}) = \begin{bmatrix} \gamma & -\gamma\bar{v}^t \\ -\gamma\bar{v} & I_3 + \frac{\gamma-1}{\bar{v}^2}\bar{v}\bar{v}^t \end{bmatrix} = \begin{bmatrix} \Lambda^{0}{}_{0} & -\Lambda^{1}{}_{0} & -\Lambda^{2}{}_{0} & -\Lambda^{3}{}_{0} \\ -\Lambda^{1}{}_{0} & & & \\ -\Lambda^{2}{}_{0} & & \left(I_3 + \frac{\Lambda^{i}{}_{0}\Lambda^{j}{}_{0}}{1+\Lambda^{0}{}_{0}}\right)_{ij} & \\ -\Lambda^{3}{}_{0} & & & \end{bmatrix}.$$

Computing then the product $\Lambda(-\bar{v})\Lambda$ we get

$$\begin{bmatrix} \Lambda^0{}_0 & -\Lambda^1{}_0 & -\Lambda^2{}_0 & -\Lambda^3{}_0 \\ -\Lambda^1{}_0 & & & \\ -\Lambda^2{}_0 & & \left(I_3 + \frac{\Lambda^i{}_0 \Lambda^j{}_0}{1+\Lambda^0{}_0}\right)_{ij} & \\ -\Lambda^3{}_0 & & & \end{bmatrix} \Lambda = \begin{bmatrix} 1 & 0 \\ \bar{z} & R \end{bmatrix}$$

for some $R$ and $\bar{z}$ since the top row is exactly given by the defining equations.

As in the $\bar{v} = 0$ case we conclude that $\bar{z} = 0$ and $R \in SO(3)$. We conclude that we have a decomposition as desired. The continuity of $\Lambda \mapsto \bar{v}$ is clear and the continuity of $\Lambda \mapsto R$ follows. □

## A.3 The Poincaré Lie Algebra Generators

Since the generators are exactly the derivatives of the one-parameter subgroups at 0, we get (by ignoring the components on which the generator are trivial, i.e. $(a, 0)$ is written as $a$.)

$$\mathbf{H}_0 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{p}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{p}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$J_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\mathbf{N}_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{N}_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{N}_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

## A.4 The Poincaré Lie Algebra Bracket

Here we calculate the Poincaré Lie algebra bracket, see Eqn. (2).

Translations commute so we have that

$$[\mathbf{p}_j, \mathbf{p}_k] = [\mathbf{p}_j, \mathbf{H}_0] = 0$$

Also translations in the $x^0$ direction commute with rotation so

$$[\mathbf{H}_0, \mathbf{p}_j] = 0$$

For an example of one of the remaining consider $[\mathbf{p}_1, J_2]$. We need to compute the product

$$\left(\begin{pmatrix} 0 \\ s \\ 0 \\ 0 \end{pmatrix}, I_4\right) \left(0, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos t & 0 & \sin t \\ 0 & 0 & 0 & 0 \\ 0 & -\sin t & 0 & \cos t \end{bmatrix}\right) \left(\begin{pmatrix} 0 \\ -s \\ 0 \\ 0 \end{pmatrix}, I_4\right)$$
$$= \left(\begin{pmatrix} 0 \\ s - s\cos t \\ 0 \\ s\sin t \end{pmatrix}, \dots\right)$$

Taking the derivative at $s = t = 0$ we get

$$\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, 0\right) = \mathbf{p}_3 = \sum_m \varepsilon_{12m} \mathbf{p}_m.$$

The rest are similar.