Proof of Equation (5.8)

Let V satisfy the assumptions of section 5.2. Equation (5.8) reads

$$\tilde{m}_\mu(\Delta) = \frac{\sqrt{\mu}}{4\pi^2}\left[\int_0^1 \frac{\sqrt{1-s}-1}{\sqrt{s^2+\left(\frac{\Delta(\sqrt{\mu})}{\mu}\right)^2}} + \frac{\sqrt{1+s}-1}{\sqrt{s^2+\left(\frac{\Delta(\sqrt{\mu})}{\mu}\right)^2}} - \frac{1}{\sqrt{1-s}} - \frac{1}{\sqrt{1+s}}\,\mathrm{d}s \right.$$

$$\left. + \int_0^1 \frac{2}{\sqrt{s^2+\left(\frac{\Delta(\sqrt{\mu})}{\mu}\right)^2}}\,\mathrm{d}s + \int_1^\infty \frac{\sqrt{1+s}}{\sqrt{s^2+\left(\frac{\Delta(\sqrt{\mu})}{\mu}\right)^2}} - \frac{1}{\sqrt{1+s}}\,\mathrm{d}s + o(1)\right].$$

To show this, it remains to be proved that

$$\lim_{\mu\to 0}\int_0^1 \frac{1}{\sqrt{s^2+x(s)^2}} - \frac{1}{\sqrt{s^2+x(0)^2}}\,\mathrm{d}s = 0,$$

where $x(s) = \frac{\Delta(\sqrt{1\pm s}\sqrt{\mu})}{\mu}$. To see this, we first show $|x(s)| \leq C|x(0)|$.

Proposition 1. *For small enough μ, the minimiser satisfies*

$$\left\|\hat{\alpha}_{\mu,V}1_{\{|p|>\varepsilon\}}\right\|_{L^{3/2}} \leq C\left\|\hat{\alpha}_{\mu,V}1_{\{|p|\leq\varepsilon\}}\right\|_{L^1}$$

for some constants $\varepsilon, C > 0$ independent of μ.

Proof. By the continuity of $\hat{V}$ we may find $\varepsilon > 0$ such that $2\hat{V}(0) \leq \hat{V}(p) \leq \frac{1}{2}\hat{V}(0) < 0$ for all $|p| \leq 2\varepsilon$. Let $\lambda = \frac{S_3}{\|V\|_{L^{3/2}}} > 1$. Then $\frac{p^2}{\lambda} + V \geq 0$ and so for the minimiser $\alpha = \alpha_{\mu,V}$ we have

$$\begin{aligned}
\mathcal{F}^{\mu,V}(\alpha) &= \frac{1}{2}\int |p^2-\mu|\left(1-\sqrt{1-4\hat{\alpha}(p)^2}\right)\mathrm{d}p + \int V(x)|\alpha(x)|^2\,\mathrm{d}x \\
&\geq \int_{|p|>\varepsilon}\left(p^2-\mu\right)\hat{\alpha}(p)^2\,\mathrm{d}p + \frac{1}{(2\pi)^{3/2}}\iint \hat{\alpha}(p)\hat{V}(p-q)\hat{\alpha}(q)\,\mathrm{d}p\,\mathrm{d}q \\
&= \int_{|p|>\varepsilon}\left(p^2-\mu\right)\hat{\alpha}(p)^2\,\mathrm{d}p + \frac{1}{(2\pi)^{3/2}}\left[\int_{|p|\leq\varepsilon}\int_{|q|\leq\varepsilon}\hat{\alpha}(p)\hat{V}(p-q)\hat{\alpha}(q)\,\mathrm{d}p\,\mathrm{d}q\right. \\
&\quad \left. +2\int_{|p|\leq\varepsilon|}\int_{|q|>\varepsilon}\hat{\alpha}(p)\hat{V}(p-q)\hat{\alpha}(q)\,\mathrm{d}p\,\mathrm{d}q + \int_{|p|>\varepsilon|}\int_{|q|>\varepsilon}\hat{\alpha}(p)\hat{V}(p-q)\hat{\alpha}(q)\,\mathrm{d}p\,\mathrm{d}q\right] \\
&\geq \left\langle\hat{\alpha}1_{\{|p|>\varepsilon\}}\middle|\frac{p^2}{\lambda}+V\middle|\hat{\alpha}1_{\{|p|>\varepsilon\}}\right\rangle + \int_{|p|>\varepsilon}\left(\left(1-\frac{1}{\lambda}\right)p^2-\mu\right)\hat{\alpha}(p)^2\,\mathrm{d}p \\
&\quad +\frac{1}{(2\pi)^{3/2}}\left[2\hat{V}(0)\left\|\hat{\alpha}1_{\{|p|\leq\varepsilon\}}\right\|_{L^1}^2 + 2\int_{|p|\leq\varepsilon}\int_{|q|>\varepsilon}\hat{\alpha}(p)\hat{V}(p-q)\hat{\alpha}(q)\,\mathrm{d}p\,\mathrm{d}q\right] \\
&\geq \int_{|p|>\varepsilon}\left(\left(1-\frac{1}{\lambda}\right)p^2-\mu\right)\hat{\alpha}(p)^2\,\mathrm{d}p \\
&\quad +\frac{1}{(2\pi)^{3/2}}\left[2\hat{V}(0)\left\|\hat{\alpha}1_{\{|p|\leq\varepsilon\}}\right\|_{L^1}^2 + 2\int_{|p|\leq\varepsilon}\int_{|q|>\varepsilon}\hat{\alpha}(p)\hat{V}(p-q)\hat{\alpha}(q)\,\mathrm{d}p\,\mathrm{d}q\right].
\end{aligned}$$

We now bound the two remaining integrals. For the first we have

$$\int_{|p|>\varepsilon}\left(\left(1-\frac{1}{\lambda}\right)p^2-\mu\right)\hat{\alpha}(p)^2\,\mathrm{d}p \geq c\int_{|p|>\varepsilon}\hat{\alpha}(p)^2\left(1+p^2\right)\,\mathrm{d}p \geq c\left\|\hat{\alpha}1_{\{|p|>\varepsilon\}}\right\|_{L^{3/2}}^2,$$

by the bound $\|\hat{g}\|_{L^{3/2}} \leq C\|g\|_{H^1}$. For the double-integral we use the Young and the Hausdorff-Young inequalities [1, Theorems 4.2 and 5.7]. We have

$$\begin{aligned}\left|\int_{|p|\leq\varepsilon}\int_{|q|>\varepsilon}\hat{\alpha}(p)\hat{V}(p-q)\hat{\alpha}(q)\,\mathrm{d}p\,\mathrm{d}q\right| &\leq C\left\|\hat{\alpha}1_{\{|p|\leq\varepsilon\}}\right\|_{L^1}\left\|\hat{V}\right\|_{L^3}\left\|\hat{\alpha}1_{\{|p|>\varepsilon\}}\right\|_{L^{3/2}}\\ &\leq C\|V\|_{L^{3/2}}\left\|\hat{\alpha}1_{\{|p|\leq\varepsilon\}}\right\|_{L^1}\left\|\hat{\alpha}1_{\{|p|>\varepsilon\}}\right\|_{L^{3/2}}.\end{aligned}$$

Combining all this we get the bound

$$\mathcal{F}^{\mu,V}(\alpha) \geq c\left\|\hat{\alpha}1_{\{|p|>\varepsilon\}}\right\|_{L^{3/2}}^2 - C_1\left\|\hat{\alpha}1_{\{|p|>\varepsilon\}}\right\|_{L^{3/2}}\left\|\hat{\alpha}1_{\{|p|\leq\varepsilon\}}\right\|_{L^1} - C_2\left\|\hat{\alpha}1_{\{|p|\leq\varepsilon\}}\right\|_{L^1}^2$$

where we absorbed the factors of V into the constants $C_1, C_2 > 0$. The right-hand-side above is a second degree polynomial in $\left\|\hat{\alpha}1_{\{|p|>\varepsilon\}}\right\|_{L^{3/2}}$. Moreover, for the minimiser $\alpha = \alpha_{\mu,V}$ we have $\mathcal{F}^{\mu,V}(\alpha) \leq 0$. We conclude that for the minimiser we have that $\left\|\hat{\alpha}1_{\{|p|>\varepsilon\}}\right\|_{L^{3/2}}$ is between the two roots of the second degree polynomial. In particular

$$\begin{aligned}\left\|\hat{\alpha}1_{\{|p|>\varepsilon\}}\right\|_{L^{3/2}} &\leq \frac{C_1\left\|\hat{\alpha}1_{\{|p|\leq\varepsilon\}}\right\|_{L^1} + \sqrt{C_1^2\left\|\hat{\alpha}1_{\{|p|\leq\varepsilon\}}\right\|_{L^1}^2 + 4cC_2\left\|\hat{\alpha}1_{\{|p|\leq\varepsilon\}}\right\|_{L^1}^2}}{2c}\\ &\leq C\left\|\hat{\alpha}1_{\{|p|\leq\varepsilon\}}\right\|_{L^1}.\end{aligned}$$

□

Now, for the function Δ we thus have

$$\begin{aligned}\Delta(p) &= \frac{2}{(2\pi)^{3/2}}\int \hat{V}(p-q)\hat{\alpha}_{\mu,V}(q)\,\mathrm{d}q\\ &= \frac{2}{(2\pi)^{3/2}}\int_{|q|\leq\varepsilon}\hat{V}(p-q)\hat{\alpha}_{\mu,V}(q)\,\mathrm{d}q + \frac{2}{(2\pi)^{3/2}}\int_{|q|>\varepsilon}\hat{V}(p-q)\hat{\alpha}_{\mu,V}(q)\,\mathrm{d}q.\end{aligned}$$

For $|p| = \sqrt{\mu}$ we have

$$\begin{aligned}|\Delta(\sqrt{\mu})| &= \frac{2}{(2\pi)^{3/2}}\int_{|q|\leq\varepsilon}|\hat{V}(\sqrt{\mu}-q)|\hat{\alpha}_{\mu,V}(q)\,\mathrm{d}q + \frac{2}{(2\pi)^{3/2}}\int_{|q|>\varepsilon}|\hat{V}(\sqrt{\mu}-q)|\hat{\alpha}_{\mu,V}(q)\,\mathrm{d}q\\ &\geq \frac{1}{(2\pi)^{3/2}}|\hat{V}(0)|\left\|\hat{\alpha}_{\mu,V}1_{\{|p|\leq\varepsilon\}}\right\|_{L^1}.\end{aligned}$$

And so, for any $|p| = \sqrt{1\pm s}\sqrt{\mu}$ we have

$$\begin{aligned}|\Delta(p)| &= \frac{2}{(2\pi)^{3/2}}\int_{|q|\leq\varepsilon}|\hat{V}(p-q)|\hat{\alpha}_{\mu,V}(q)\,\mathrm{d}q + \frac{2}{(2\pi)^{3/2}}\int_{|q|>\varepsilon}|\hat{V}(p-q)|\hat{\alpha}_{\mu,V}(q)\,\mathrm{d}q\\ &\leq \frac{4}{(2\pi)^{3/2}}|\hat{V}(0)|\left\|\hat{\alpha}_{\mu,V}1_{\{|p|\leq\varepsilon\}}\right\|_{L^1} + \frac{2}{(2\pi)^{3/2}}\left\|\hat{V}\right\|_{L^3}\left\|\hat{\alpha}_{\mu,V}1_{\{|p|>\varepsilon\}}\right\|_{L^{3/2}}\\ &\leq C\left\|\hat{\alpha}_{\mu,V}1_{\{|p|\leq\varepsilon\}}\right\|_{L^1} \leq C|\Delta(\sqrt{\mu})|,\end{aligned}$$

by the Hausdorff-Young inequality [1, Theorem 5.7] and the bound above. For the function $x(s)$ we thus have $|x(s)| \leq C|x(0)|$. As already noted in the thesis, this proves the desired. For convenience we give the argument here as well.

With the Lipschitz bound on Δ we have that $|x(s) - x(0)| \leq C\mu^{1/4}s$. Hence

$$
\begin{aligned}
\left| \frac{1}{\sqrt{s^2 + x(s)^2}} - \frac{1}{\sqrt{s^2 + x(0)^2}} \right| &= \frac{\left| x(s)^2 - x(0)^2 \right|}{\sqrt{s^2 + x(s)^2}\sqrt{s^2 + x(0)^2}\left(\sqrt{s^2 + x(s)^2} + \sqrt{s^2 + x(0)^2}\right)} \\
&\leq \frac{C\mu^{1/4} s |x(0)|}{\sqrt{s^2 + x(s)^2}\sqrt{s^2 + x(0)^2}\left(s + \sqrt{s^2 + x(0)^2}\right)} \\
&\leq C\mu^{1/4} \frac{|x(0)|}{\sqrt{s^2 + x(0)^2}\left(s + \sqrt{s^2 + x(0)^2}\right)}.
\end{aligned}
$$

Now, one may compute that

$$
\int_0^1 \frac{|x(0)|}{\sqrt{s^2 + x(0)^2}\left(s + \sqrt{s^2 + x(0)^2}\right)} \, \mathrm{d}s = O(1).
$$

This shows that

$$
\int_0^1 \frac{1}{\sqrt{s^2 + x(s)^2}} - \frac{1}{\sqrt{s^2 + x(0)^2}} \, \mathrm{d}s = O\left(\mu^{1/4}\right)
$$

vanishes as desired.

References

[1] E. H. Lieb and M. Loss. *Analysis*. Graduate studies in mathematics ; 14. American Mathematical Society, Providence, R.I, 2. ed. edition, 2001. ISBN 0821827839.