Course material overview — Quantum Mechanics

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Abstract

Overview of lectures, exercises, and problem sets for the course "Quantum Mechanics" taught at Université Paris-Dauphine in Autumn 2025. The course ran for 13 weeks with 1.5 hours of lectures and 1.5 hours of tutorials each week.

Many of the exercises/problems are simply copied from (or slightly adapted from) [GS18].

References

[DS14] B. Durhuus and J. P. Solovej. *Mathematical Physics*. Lecture notes. 2014. URL: `https://noter.math.ku.dk/mathphys2014.pdf`.

[GS18] D. J. Griffiths and D. F. Schroeter. *Introduction to Quantum Mechanics*. 3rd edition. Cambridge University Press, 2018. DOI: `10.1017/9781316995433`.

1 Lectures

5/9 Wave functions, position, momentum, Schrödinger equation, uncertainty relation, time-independent Schrödinger equation. (Griffiths sections 1.1, 1.2, 1.4, 1.5, 1.6, 2.1) [2x lecture]

9/9 Time-independent Schrödinger equation, particle in a square well. (Griffiths sections 2.1, 2.2)

16/9 Free particles, δ-potential. (Griffiths sections 2.4, 2.5)

23/9 Hilbert spaces. (Durhuus–Solovej sections 4.1, 4.2, 4.3, 4.4)

30/9 Hilbert spaces, operators. (Durhuus–Solovej sections 4.4, 4.5, 4.6, 5.3)

3/10 Operators on Hilbert spaces. (Durhuus–Solovej sections 5.3, 5.4, 5.5)

14/10 Postulates of quantum mechanics, the uncertainty relation. (Griffiths sections 3.1, 3.2, 3.3, 3.4, 3.5)

4/11 The harmonic oscillator, the time-energy uncertainty principle. (Griffiths sections 2.3, 3.5)

14/11 Angular momentum. (Griffiths section 4.3)

21/11 The Hydrogen atom, spin. (Griffiths sections 4.2, 4.4) [2x lecture]

5/12 Addition of angular momentum, multiple particles. (Griffiths sections 4.4, 5.1)

12/12 Atoms. (Griffiths section 5.2)

2 Exercises

2.1 Week of 2025-09-08

Exercise 2.1.1. Consider the wave function $\Psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t}$ for $A, \lambda, \omega > 0$ constants.

(a) Find A such that Ψ is normalized.

(b) Determine $\langle \hat{x} \rangle$ and $\langle \hat{x}^2 \rangle$

(c) Make a sketch of $|\Psi(x,t)|^2$

Exercise 2.1.2. A particle of mass m has wave function $\Psi(x,t) = Ae^{-\omega m x^2/2\hbar}e^{-i\omega t}$ with $\omega, A > 0$

(a) Find A such that Ψ is normalized

(b) For what potential function $V(x)$ is Ψ a solution to the Schrödinger equation?

(c) Calculate $\langle \Delta\hat{x}^2 \rangle$ and $\langle \Delta\hat{p}^2 \rangle$. Compare to the uncertainty relation.

Exercise 2.1.3. Consider the Hamiltonian operator $\hat{H} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \hat{V}$. Show that $\frac{\mathrm{d}\langle \hat{H} \rangle}{\mathrm{d}t} = 0$.

Exercise 2.1.4. Show that for any solution ψ to the time-independent Schrödinger equation $\hat{H}\psi = E\psi$ we must have E real. *Hint:* Show first that $E = \left\langle \hat{H} \right\rangle$ and use integration by parts afterwards.

Exercise 2.1.5. Prove that for any wave function we have $\left\langle \hat{H} \right\rangle \geq \min_x V(x)$.

Exercise 2.1.6. Prove that the time-independent wave function ψ may always be taken real. (This is not the same as all such wave functions being real, only that any complex-valued one can be written as a linear combination of real-valued ones of the same energy.) *Hint:* If ψ satisfies the time-independent Schrödinger equation, so does $\overline{\psi}$.

2.2 Week of 2025-09-15

Exercise 2.2.1. Recall that an operator $\hat{A}$ is Hermitian if $\int \overline{\psi}\hat{A}\phi \,\mathrm{d}x = \int \overline{\hat{A}\psi}\phi \,\mathrm{d}x$ for any two wave functions ψ and ϕ. (Whether ψ and ϕ are normalized do not matter. Convince yourself of this.)

(a) Show that $\hat{x}$ and $\hat{p}$ are Hermitian operators.

(b) Suppose that $\hat{A}$ is Hermitian. Show that $\hat{A}^n$ is Hermitian.

(c) Let $\hat{A}$ and $\hat{B}$ be Hermitian. Determine for each of the following operators if they are Hermitian:

$$i\hat{A}, \qquad \hat{A}\hat{B}, \qquad [\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}, \qquad i[\hat{A},\hat{B}].$$

Compare to what you know about Hermitian matrices.

Exercise 2.2.2. A particle in an infinite square well is initialized in an even mixture of the first two states:

$$\Psi(x, t = 0) = A(\psi_1(x) + \psi_2(x))$$

(a) Find A such that Ψ is normalized (*Hint:* Use the orthonormality of ψ_1 and ψ_2.)

(b) Compute $\Psi(x,t)$ and $|\Psi(x,t)|^2$.

(c) If you were to measure the energy of the particle, what values are possible? With what probability would you measure them?

(d) Compute $\langle x \rangle$. It oscillates in time. With what frequency? Amplitude?

(e) Compute $\langle p \rangle$.

Exercise 2.2.3. For each stationary state ψ_n of an infinite square well. Compute $\langle \hat{x} \rangle, \langle \hat{x}^2 \rangle, \langle \hat{p} \rangle, \langle \hat{p}^2 \rangle$. Verify that the uncertainty relation is satisfied.

Exercise 2.2.4. Recall that the δ-function is the "function" $\delta(x)$ with the property that

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,\mathrm{d}x = f(0)$$

for any function f. This is not a proper function (you cannot write down the values $\delta(x)$) so it must be treated with some care. It is nonetheless very important and useful in physics.

Calculate the following

(a) $\int_{-3}^{4} xe^x\delta(x-2)\,\mathrm{d}x$

(b) $\int_{-7}^{4} e^x\delta(|x|-2)\,\mathrm{d}x$

(c) $\int_{-\infty}^{\infty} f(x)\delta(2x)\,\mathrm{d}x$. Conclude that $\delta(2x) = c\delta(x)$. What is c?

(d) $\int_{\infty}^{x} \delta(y)\,\mathrm{d}y$. What is then $f'(x)$ for $f = \chi_{[0,1]}$ (the indicator function of the interval $[0,1]$)?

Exercise 2.2.5. Consider a free particle initialized in the state $\Psi(x, t = 0) = Ae^{-\alpha x^2}$ for some $A, \alpha > 0$.

1. Find A such that Ψ is normalized.
2. Find $\Psi(x,t)$. *Hint:* It may prove helpful that

$$\int_{-\infty}^{\infty} e^{-x^2} e^{-ikx}\,\mathrm{d}x = \sqrt{\pi}e^{-k^2/4}.$$

3. Find $|\Psi(x,t)|^2$. Sketch both $|\Psi(x,t=0)|^2$ and $|\Psi(x,t)|^2$ for some large t.

Exercise 2.2.6. Consider scattering off a finite well ($V_0 > 0$):

$$V(x) = \begin{cases} -V_0 & \text{if } |x| \leq a \\ 0 & \text{if } |x| > a. \end{cases}$$

Consider the stationary solution of the Schrödinger equation with energy $E > 0$. Then, solutions are of the form

$$\psi(x) = \begin{cases} e^{ikx} + Ae^{-ikx} & x \leq -a \\ Be^{iqx} + Ce^{-iqx} & -a < x \leq a \\ De^{ikx} & x > a \end{cases}$$

for appropriate $k, q > 0$ and $A, B, C, D \in \mathbb{C}$.

(a) Determine k, q.

(b) Write down the equations on A, B, C, D coming from continuity of ψ and ψ'.

(c) Determine the Transmission coefficient $T = |D|^2$. Draw a sketch of T as a function of the energy E.

Note: This problem and its solution is the contents of Griffiths section 2.6. Work on it yourself before looking in the book.

2.3 Week of 2025-09-22

Exercise 2.3.1. Evaluate the following

(a) $\int_{-2}^{2}(2x+3)\delta(x)\,\mathrm{d}x$

(b) $\int_{-2}^{2}(2x+3)\delta(3x)\,\mathrm{d}x$

(c) $\int_{0}^{2}(x^3+3x+2)\delta(1-x)\,\mathrm{d}x$

(d) $\int_{-\infty}^{a}\delta(x-b)\,\mathrm{d}x$

Exercise 2.3.2. Show that

(a) $f(x)\delta(x) = f(0)\delta(x)$. In particular, $\delta(x)\mathbb{1}_{(a,b)} = 0$ for $0 < a < b$.

(b) $x\frac{\mathrm{d}\delta(x)}{\mathrm{d}x} = -\delta(x)$.

(c) $\frac{\mathrm{d}\theta(x)}{\mathrm{d}x} = \delta(x)$, where $\theta(x) = \mathbb{1}_{[0,\infty)}$ is called the *Heaviside* function.

Hint: Two expressions involving δ-functions are equal if their integrals against any test function are equal.

Exercise 2.3.3. Compute the Fourier transform of $\delta(x)$. Computing next the inverse Fourier transform of the result suggests that

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,\mathrm{d}k$$

Prove this relation by considering their integrals against a test function. *Hint: Note that $e^0 = 1$, what does this say about the Fourier transform of a function evaluated at $k = 0$?*

Exercise 2.3.4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function with $f(x_0) = 0$ for some x_0. Compute $\delta(f(x))$.

Exercise 2.3.5. Consider the interaction $V(x) = \alpha\delta(x)$. Determine the transmission and reflection coefficients as a function of the energy.

Exercises 4.4, 4.9, 4.11 in Durhuus–Solovej

2.4 Week of 2025-10-06 (2x tutorial)

Exercise 2.4.1. Show that a norm $\|\cdot\|$ is given by an inner product if and only if it satisfies the *parallelogram law*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Hint: Use the polarization identity.

Exercise 2.4.2. Let $\{e_n\}_{n=0}^{\infty}$ be the standard basis for $\ell^2(\mathbb{N})$; i.e., $e_n = (0,\dots,0,1,0,\dots)$ with a 1 in the n'th entry. Identify $\operatorname{span}\{e_n : n = 0,1,\dots\}$ and show that $\{e_n\}_{n=0}^{\infty}$ is indeed a basis for $\ell^2(\mathbb{N})$, i.e., show that $\overline{\operatorname{span}\{e_n : n = 0,1,\dots\}} = \ell^2(\mathbb{N})$.

Exercise 2.4.3. Let A, B be two commuting hermitian matrices. Show that A and B can be diagonalized simultaneously, i.e. that there exists a basis consisting of vectors, which are eigenvectors of both A and B.

Hint: Use the spectral theorem to first diagonalize A. What can you say about the action of B on an eigenspace of A?

Remark: This result generalizes to bounded self-adjoint operators, when diagonalizability is understood in an appropriate generalized sense.

Exercise 2.4.4. Show the converse of the above exercise. That is, if A and B are hermitian matrices, such that there exists a basis of eigenvectors of both A and B, then A and B commute.

Exercise 2.4.5. By the spectral theorem for Hilbert–Schmidt operators (Thm. 5.17), every self-adjoint Hilbert–Schmidt operator is diagonalizable. Show that the converse fails, i.e., give an example of a diagonalizable operator, that is not Hilbert–Schmidt.

Exercise 2.4.6. Let H_1, H_2 be Hilbert spaces and $A : H_1 \to H_2$ be a bounded operator, that is, $\|A\| < \infty$, where the norm is defined by

$$\|A\| = \sup\{\|Ax\| : \|x\| = 1\}.$$

Prove that

$$\|A\| = \sup\{|\langle y|Ax\rangle| : \|x\|, \|y\| = 1\} = \sup\{\operatorname{Re}\langle y|Ax\rangle : \|x\|, \|y\| = 1\}.$$

Exercise 2.4.7. Let H_1, H_2, H_3 be Hilbert spaces, $A, B : H_1 \to H_2$ and $C : H_2 \to H_3$ be bounded operators, and $\lambda \in \mathbb{C}$. Prove that

$$\begin{aligned} \|A\| &\geq 0 \text{ with } = \text{ if and only if } A = 0, \\ \|A^*\| &= \|A\|, \\ \|\lambda A\| &= |\lambda|\,\|A\|, \\ \|A+B\| &\leq \|A\| + \|B\|, \\ \|CA\| &\leq \|C\|\,\|A\|. \end{aligned}$$

Exercises 4.13, 4.14, 4.16, 4.18, 4.19, 5.7, 5.9, 5.11, 5.16, 5.20 in Durhuus–Solovej

2.5 Week of 2025-10-13

Exercise 2.5.1. Cite a Hamiltonian with

(a) only discrete spectrum,

(b) only continuous spectrum (other than the free particle),

(c) both discrete and continuous spectrum.

Recall that discrete spectrum means there exists associated eigenfunctions, and continuous spectrum means there exists associated generalized (meaning non-normalizable) eigenfunctions.

Exercise 2.5.2. Consider the operator $A = \frac{\mathrm{d}}{\mathrm{d}x} + cx$ on $L^2(\mathbb{R})$ for some constant $c > 0$.

(a) Determine the adjoint $A^\dagger$,

(b) Compute the operator $A^\dagger A$,

(c) Compute the commutator $[A, A^\dagger] = AA^\dagger - A^\dagger A$.

Exercise 2.5.3 (Solve 5.12, 5.13, 5.14 in Durhuus–Solovej first)**.** Let H be a Hamiltonian operator and suppose that we can diagonalize H. That is, we have $H = \sum_{n=0}^{\infty} E_n |\psi_n\rangle\langle\psi_n|$, with $\{\psi_n\}_{n=0}^{\infty}$ an orthonormal set.

(a) Use Ex 5.14 to see that $\Psi(x,t) = e^{-itH/\hbar}\Psi(x, t=0)$ solves the Schrödinger equation.

(b) Using the formula from Ex 5.12, rewrite the operator $e^{-itH/\hbar}$ in terms of the diagonalization of H.

(c) Find a formula for $\Psi(x,t)$. Does this remind you of anything?

Exercises 5.12, 5.13, 5.14, 5.18, 5.19 in Durhuus–Solovej

2.6 Week of 2025-11-03

Exercise 2.6.1. Let H be a Hilbert space, $\psi \in H$ a vector, $A \in B(H)$ a bounded operator and $A^\dagger$ its adjoint. Show that $\langle\psi| A^\dagger = \langle A\psi|$.

Exercise 2.6.2. Let P be an orthogonal projection on a Hilbert space H. Show that P is *idempotent*, i.e., $P^2 = P$. What are the eigenvalues of P?

Exercise 2.6.3. An operator $\hat{A}$ representing an observable A has eigenstates ψ_1, ψ_2 with eigenvalues a_1, a_2 respectively. Additionally, an operator $\hat{B}$ representing an observable B has eigenstates ϕ_1, ϕ_2 with eigenvalues b_1, b_2. Suppose that

$$\psi_1 = \frac{3\phi_1 + 4\phi_2}{5}, \qquad \psi_2 = \frac{4\phi_1 - 3\phi_2}{5}$$

(a) Observable A is measured yielding the outcome a_1. What is the state of the system immediately after this measurement?

(b) If we subsequently were to measure B, what are the possible outcomes, and what are the associated probabilities?

(c) After B is measured, we measure again A. What is the probability that the measurement of A yields a_1?

Exercise 2.6.4. Let $\Psi(x,t)$ be a wave function and let $\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int \Psi(x,t) e^{-ipx/\hbar}\, \mathrm{d}x$ denote its Fourier transform. Show that

$$\langle \hat{x} \rangle = \int \overline{\Phi(p,t)} \left(i\hbar \frac{\partial}{\partial p} \right) \Phi(p,t)\, \mathrm{d}p, \qquad \langle \hat{p} \rangle = \int p |\Phi(p,t)|^2\, \mathrm{d}p.$$

This illustrates that, in Fourier variables, $\hat{p}$ is a multiplication operator (multiplication by p) and $\hat{x} = i\hbar\frac{\partial}{\partial p}$ is a differential operator.

Exercise 2.6.5. (a) Show for any operators A, B, C the identity $[AB, C] = A[B,C] + [A,C]B$,

(b) Show that $[x^n, p] = i\hbar n x^{n-1}$,

(c) Show that $[f(x), p] = i\hbar \frac{\mathrm{d}f(x)}{\mathrm{d}x}$ for any function f.

Exercise 2.6.6. Show that for any function f and any x_0 we have

$$f(x + x_0) = e^{i\hat{p}x_0/\hbar} f(x)$$

Hint: Use Fourier transforms.

Exercise 2.6.7 (Tunneling)**.** Consider the double δ-well potential from problem set #1, $V(x) = -\alpha\delta(x-a) - \alpha\delta(x+a)$. For a sufficiently large, we saw that there were two bound states:

$$\psi_{\text{even}}(x) = \begin{cases} A_1 e^{k_1 x} & x < -a, \\ B_1 \left(e^{k_1 x} + e^{-k_1 x}\right) & -a < x < a, \\ A_1 e^{-k_1 x} & x > a, \end{cases} \qquad \psi_{\text{odd}}(x) = \begin{cases} -A_2 e^{k_2 x} & x < -a, \\ B_2 \left(-e^{k_2 x} + e^{-k_2 x}\right) & -a < x < a, \\ A_2 e^{-k_2 x} & x > a. \end{cases}$$

For a large we have $A_1 \approx A_2 \approx B_1 \approx B_2$ and $k_1 \approx k_2$. (But importantly $k_1 \neq k_2$ are not exactly equal.)

Consider a particle initially in state $\psi_{\text{right}} = \frac{1}{\sqrt{2}}(\psi_{\text{even}} + \psi_{\text{odd}})$, which is essentially localized near the right δ-well. Show that, after some time T, the particle is instead in the state $\psi_{\text{left}} = \frac{1}{\sqrt{2}}(\psi_{\text{even}} - \psi_{\text{odd}})$, which is localized near the left well.

2.7 Week of 2025-11-17

Exercise 2.7.1. Let a_- and a_+ denote the lowering and raising ladder operators for the harmonic oscillator. Show that they are each other's adjoints. That is, $a_-^\dagger = a_+$ and $a_+^\dagger = a_-$.

Exercise 2.7.2. Compute $\langle x \rangle, \langle x^2 \rangle, \langle p \rangle, \langle p^2 \rangle$ and $\sigma_x \sigma_p$ in the n'th state of the harmonic oscillator.
Compare $\langle T \rangle$ and $\langle V \rangle$ (with $T = \frac{p^2}{2m}$ the kinetic energy) with the total energy $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$.
Hint: Use that $x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)$ and $p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$.

Exercise 2.7.3. Show that the uncertainty relation can be strengthened to

$$\sigma_A^2 \sigma_B^2 \geq \frac{1}{4}\left(|\langle AB + BA \rangle - 2\langle A \rangle \langle B \rangle|^2 + |\langle [A,B] \rangle|^2 \right).$$

Hint: Redo the proof of the uncertainty relation, but do not throw away $\operatorname{Re} z$.

Exercise 2.7.4. Find the matrix elements of the operators x and p in the basis of eigenstates of the harmonic oscillator. That is, calculate $\langle n| x |n'\rangle$ and $\langle n| p |n'\rangle$ for $|n\rangle = \psi_n$, the n'th eigenstate of the harmonic oscillator.

Show that for the corresponding (infinite) matrices X, P we have that $\frac{1}{2m}P^2 + \frac{m\omega^2}{2}X^2$ is a diagonal (infinite) matrix. Are its diagonal entries as you would expect?

Exercise 2.7.5. Show that $L_+^\dagger = L_-$ and $L_-^\dagger = L_+$ for the raising and lowering operators for angular momentum.

Exercise 2.7.6. Suppose that V is a radial function. Show that $[L^2, H] = [L_z, H] = 0$.

Exercise 2.7.7. Show that $\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{L}\rangle = \langle \mathbf{r} \times (-\nabla V)\rangle$, where $\mathbf{L} = (L_x, L_y, L_z)$. (This equation should be understood component-wise.) Conclude for a radial potential V that $\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{L}\rangle = 0$.
Note: This is the rotational analogue of Ehrenfest's theorem: Change of angular momentum is given by the torque $\mathbf{r} \times \mathbf{F} = \mathbf{r} \times (-\nabla V)$. For radial potentials we have "conservation of angular momentum".

Exercise 2.7.8. Recall the formula for the gradient in spherical coordinates (physicists convention: θ is the angle to the pole. Hats denote unit vectors in their respective directions)

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}, \qquad \hat{r} = (\sin\theta\cos\phi)\hat{x} + (\sin\theta\sin\phi)\hat{y} + (\cos\phi)\hat{z},$$
$$\hat{\theta} = (\cos\theta\cos\phi)\hat{x} + (\cos\theta\sin\phi)\hat{y} - (\sin\theta)\hat{z}, \quad \hat{\phi} = -(\sin\phi)\hat{x} + (\cos\phi)\hat{y}.$$

Derive expressions for L_z, L^2 in spherical coordinates.
Hint: To derive the expression for L^2, derive it first for $L_\pm$ and L_z and use that $L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$.
Hint: Part of this calculation is explained in Griffiths section 4.3.

2.8 Week of 2025-12-01 (2x tutorial)

Exercise 2.8.1. Determine the coefficients $A^\pm_{lm}$ such that $L_\pm |lm\rangle = A^\pm_{l(m\pm1)} |l(m\pm 1)\rangle$.

Hint: Use the formula for L^2 in terms of $L_\pm$ and L_z.

Exercise 2.8.2. Find the wave functions ψ_{nlm} for $n = 2$, $l = 0, 1$ and $m = -l, \ldots, l$

Hint: To find the angular part, either read Griffiths section 4.1, or look up the formulas in some table.

Exercise 2.8.3. Determine the probability distribution of the radial coordinate in the ground state of Hydrogen. That is, determine the probability that the electron would be found between radii r and $r + \mathrm{d}r$.

Exercise 2.8.4. A *hydrogenic* atom consists of a single electron and a nucleus of Z protons, i.e. $Z = 2$ corresponds to ionized Helium and $Z = 3$ double ionized Lithium and so on. Determine the bound state energies $E_n(Z)$ and Bohr radius $a_0(Z)$.

Exercise 2.8.5. Show that the Pauli spin matrices satisfies

$$\sigma_j\sigma_k = \delta_{jk}\mathbb{1} + i\sum_l \epsilon_{jkl}\sigma_l, \qquad [\sigma_j, \sigma_k] = 2i\sum_l \epsilon_{jkl}\sigma_l, \qquad \sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}\mathbb{1}$$

where ϵ_{jkl} is the *Levi–Civita* symbol: $\epsilon_{jkl} = 1$ if $jkl = 123, 231, 312$ and $\epsilon_{jkl} = -1$ if $jkl = 132, 213, 321$ and $\epsilon_{jkl} = 0$ otherwise. (Here the indices are to be understood as $1 = x$, $2 = y$ and $3 = z$).

Exercise 2.8.6. Suppose a spin-1/2 particle is in an eigenstate of S_z. What are the possible outcomes when measuring the spin in the x-direction? With what probabilities do we measure them?

Exercise 2.8.7. Consider an electron in the time-*dependent* magnetic field

$$\mathbf{B} = B_0\cos(\omega t)\hat{z},$$

where B_0 and ω are constants.

(a) What is the Hamiltonian describing the system?

(b) Suppose that at time $t = 0$ the electron is in the spin-up state with respect to the x-axis. Determine the state $\chi(t)$.

Note: This is a time-dependent Hamiltonian, so you cannot just apply the general theory we have used in the course. You can, however, solve the time-dependent Schrödinger equation directly.

(c) What is the probability of measuring $-\hbar/2$, when measuring S_x?

Exercise 2.8.8. Show that any hermitian 2×2 matrix can be written as a $\mathbb{R}$-linear combination of $\sigma_x, \sigma_y, \sigma_z$ and the identity matrix $\mathbb{1}$.

Exercise 2.8.9. Compute $\exp(ia\hat{\mathbf{n}} \cdot \boldsymbol{\sigma})$, where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices, and $\mathbf{n} \in \mathbb{S}^2$ is a unit vector.

Hint: For the power series approach, it may prove helpful to study separately the even and odd powers of $\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}$.

Exercise 2.8.10 (Bloch Sphere)**.** All states on the Hilbert space $\mathbb{C}^2$ can be mapped to points on a sphere, called the *Bloch Sphere*. Let $|0\rangle, |1\rangle$ denote an orthonormal basis of $\mathbb{C}^2$. For any state $|\psi\rangle$ we can write

$$|\psi\rangle = e^{i\alpha}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right), \qquad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi$$

This defines a point on the 2-sphere $\mathbf{a} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \in \mathbb{S}^2$.

(a) Show that the map $|\psi\rangle \mapsto \mathbf{a}$ is surjective, and show that, modulo the arbitrary phase $e^{i\alpha}$, it is also injective.

(b) Show that $|\psi\rangle\langle\psi| = \frac{1}{2}(\mathbb{1} + \mathbf{a} \cdot \boldsymbol{\sigma})$, where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices.

Exercise 2.8.11. What is the probability that an electron in the Hydrogen ground state is found inside the nucleus?

(a) Let b denote the radius of the nucleus and determine the result as a Taylor expansion to lowest order in b/a_0.

(b) Pretend that $\psi(r)$ is essentially constant $\approx \psi(0)$ over the domain of the nucleus such that the probability is $P \approx 4\pi b^3|\psi(0)|^2/3$. Do you get the same result?

2.9 Week of 2025-12-08

Exercise 2.9.1. Consider the *three-dimensional* infinite square (cubical) well

$$V(x,y,z) = \begin{cases} 0 & \text{if } 0 \leq x,y,z \leq L, \\ +\infty & \text{otherwise.} \end{cases}$$

Use separation of variables in Cartesian coordinates and find the energies and their degeneracies.

Exercise 2.9.2. Verify that the singlet and triplet states are eigenstates of the spin operators with the appropriate eigenvalues. That is, compute S^2 applied to the states

$$\left\{\begin{aligned} |1\ 1\rangle &= |\uparrow\uparrow\rangle \\ |1\ 0\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |1\ {-1}\rangle &= |\downarrow\downarrow\rangle \end{aligned}\right\} \quad \text{(triplet)}, \qquad \left\{|0\ 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)\right\} \quad \text{(singlet)}.$$

Exercise 2.9.3. Using problem 5 from the last problem set, correct our analysis of the Hydrogen atom taking into account the motion of the nucleus. (You don't need to have solved the problem to do this exercise.)

(a) What is the new binding energy (ground state energy)? Plug in numerical values for the mass of the electron and proton.

(b) Consider *positronium*, in which the proton is replaced by a positron (same mass as the electron, but positive charge $+e$). What is the binding energy?

Exercise 2.9.4. Consider two identical fermions in an infinite square well. Determine the energies and their degeneracies. (Ignore spin. If you don't like this, consider the particles to be in the same spin-state.)

Exercise 2.9.5. For two identical particles in the states ψ_a and ψ_b the bosonic and fermionic two-particle wave functions are

$$\psi_+(x_1,x_2) = A_+(\psi_a(x_1)\psi_b(x_2) + \psi_a(x_2)\psi_b(x_1)), \qquad \psi_-(x_1,x_2) = A_-(\psi_a(x_1)\psi_b(x_2) - \psi_a(x_2)\psi_b(x_1)),$$

Suppose that ψ_a, ψ_b are normalized. Determine $A_\pm$ such that $\psi_\pm$ are normalized.

Exercise 2.9.6. Consider three particles in states ψ_a, ψ_b and ψ_c. Suppose that ψ_a, ψ_b and ψ_c are orthonormal. Construct the three-particle wave functions $\psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ in the cases

(a) distinguishable particles,

(b) identical bosons,

(c) identical fermions.

Keep in mind that, in the bosonic/fermionic case, $\psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ must be symmetric/antisymmetric under exchange of *any* pair of particles.

Hint: For the fermionic state, consider the *Slater determinant*, the determinant of the matrix with entries $\psi_X(\mathbf{r}_j)$ for $X \in \{a, b, c\}$ and $j \in \{1, 2, 3\}$. This construction works for an arbitrary number of particles.

Exercise 2.9.7. Consider N non-interacting identical particles in the one-dimensional infinite square well. Consider the limit $N \to \infty$ and $L \to \infty$ such that the *density* of particles $n = N/L$ is constant. Determine the ground state energy per particle in the case of the particles being

(a) Bosons

(b) Fermions (ignoring spin)

Remark: For a slightly more computationally difficult exercise, consider the three-dimensional case. The solution is explained in Griffith section 5.3 "solids".

3 Homework problems

3.1 Problem Sheet #1 —— Hand in by Friday 3 October

Problem 3.1.1. Show that $\frac{\mathrm{d}\langle \hat{p} \rangle}{\mathrm{d}t} = \left\langle -\frac{\mathrm{d}\hat{V}}{\mathrm{d}x} \right\rangle$.

Problem 3.1.2. Assume that $V(x)$ is an even function. (That is, $V(-x) = V(x)$ for all x.) Prove that the time-independent wave function ψ may always be taken either even or odd. *Hint:* If $\psi(x)$ satisfies the time-independent Schrödinger equation, so does $\psi(-x)$.

Problem 3.1.3. Suppose a wave function $\psi(x)$ has

$$\langle \hat{x} \rangle = y, \qquad \langle \Delta\hat{x}^2 \rangle = \sigma_x^2, \qquad \langle \hat{p} \rangle = q, \qquad \langle \Delta\hat{p}^2 \rangle = \sigma_p^2.$$

Determine $\langle \hat{x} \rangle, \langle \Delta\hat{x}^2 \rangle, \langle \hat{p} \rangle$, and $\langle \Delta\hat{p}^2 \rangle$ for the wave function $\tilde{\psi}(x) = \psi(x - x_0)e^{ip_0x/\hbar}$.

Problem 3.1.4. A particle in an infinite square well is initialized with wave function

$$\Psi(x, t = 0) = \begin{cases} Ax & 0 \leq x \leq L/2 \\ A(L - x) & L/2 \leq x \leq L \end{cases}$$

(a) Find A such that Ψ is normalized and make a sketch of $\Psi(x, t = 0)$.

(b) Determine $\Psi(x, t)$ and the expectation value of the energy.

Problem 3.1.5. Show that the wave function of a particle in an infinite square well returns to its original state after a revival time $T = \frac{4mL^2}{\pi\hbar}$. That is, $\Psi(x, T) = \Psi(x, 0)$ for any state.

What is the analogous *classical* revival time for a particle of energy E bouncing elastically in a well?

Problem 3.1.6. Consider the potential $V(x) = -\alpha\delta(x + a) - \alpha\delta(x - a)$ for some $a, \alpha > 0$.

(a) Find all possible bound states and their energies. How many are there? *Hint: V is an even function.*

You may use that

$$\frac{z}{1 + e^{-z}} = t \quad \text{has a unique solution } z > 0 \text{ for any } t > 0\text{, and}$$
$$\frac{z}{1 - e^{-z}} = t \quad \text{has a unique solution } z > 0 \text{ for any } t > 1\text{, but no solution } z > 0 \text{ for } t \leq 1.$$

(b) Sketch the bound state(s) for $a = \hbar^2/4m\alpha$ and $a = \hbar^2/m\alpha$. Do they behave as expected for a large?

(c) What is the transmission coefficient for scattering with energy $E > 0$?

Problem 3.1.7. Consider the Hilbert space $L^2([-a, a])$. Let M denote the subspace of all even functions. Find an orthonormal basis for M. *Hint: What do the Fourier series of an even function look like?*

Problem Sheet #2 —— Hand in by Friday 21 November

Problem 3.1.8. Show that $\sigma_x \sigma_H \geq \frac{\hbar}{2m} |\langle \hat{p} \rangle|$.

Problem 3.1.9. Let ψ be any (normalized) wave function and let $\mathcal{F}\psi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int \psi(x) e^{-ipx/\hbar} \, dx$ denote the Fourier transform of ψ.

(a) Determine the Fourier transform of the functions $x^n \psi(x)$ and $\psi^{(n)}(x)$ for any $n \in \mathbb{N}$.

(b) Let P be a random variable with law $|\mathcal{F}\psi(p)|^2 \, dp$. Show that the moments of P satisfy $\mathbb{E}P^n = \langle \hat{p}^n \rangle$ for all $n \in \mathbb{N}$.

(c) Let $f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$ be the generalized eigenfunctions of the momentum operator. Show that for any analytic function g we have

$$g(\hat{p}) = \int g(p) \, |f_p\rangle \langle f_p| \, dp.$$

Problem 3.1.10 (Qubit)**.** Suppose a quantum system has state space (Hilbert space) $\mathbb{C}^2$. Up to a constant shift of the energy, any Hamiltonian on this space can be written as

$$H = \begin{pmatrix} V & K \\ \overline{K} & -V \end{pmatrix}, \qquad V \in \mathbb{R}, \quad K \in \mathbb{C}.$$

Suppose a particle is initialized in the state $\psi(t=0) = \psi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. For the time-evolved state $\psi(t)$ determine the overlap $|\langle \psi_0 | \psi(t) \rangle|^2$. Does your result behave as expected for $K = 0$ or $V = 0$?

Problem 3.1.11. Let $|n\rangle$ denote the n'th eigenstate of the harmonic oscillator. Define $U : L^2(\mathbb{R}) \to \ell^2(\mathbb{N})$ by

$$U |n\rangle = e_n = (0, \dots, 0, 1, 0, \dots), \qquad \text{(with the 1 in the } n\text{'th entry)}$$

and extended by linearity. Show that U defines a unitary operator and determine the operator $U a_- U^*$.
Hint: You may use that the eigenstates of the harmonic oscillator form an orthonormal basis of $L^2(\mathbb{R})$.

Problem 3.1.12 (Coherent states)**.** A particularly nice class of states for the harmonic oscillator are the *coherent states*. They are the eigenfunctions of the lowering operator, and we denote them by $|\alpha\rangle$ for any complex number α:

$$a_- |\alpha\rangle = \alpha |\alpha\rangle . \tag{1}$$

Note that there exists such a coherent state for any eigenvalue $\alpha \in \mathbb{C}$.

(a) Calculate $\langle x \rangle, \langle x^2 \rangle, \langle p \rangle, \langle p^2 \rangle$ in the state $|\alpha\rangle$ and show that $\sigma_x \sigma_p = \frac{\hbar}{2}$.

Hint: Use that $x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)$ and $p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$.

(b) Expanding $|\alpha\rangle$ in the basis of eigenfunctions of the harmonic oscillator, show that

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle , \qquad c_n = \frac{\alpha^n}{\sqrt{n!}} \exp\left(-\frac{|\alpha|^2}{2}\right).$$

Note: Here $|n\rangle$ refers to the n'th eigenstate of the harmonic oscillator and not a coherent state of eigenvalue n.

Hint: Find first c_n in terms of c_0.

(c) Consider the time-evolved state $\Psi(t,x)$ with initial condition $\Psi(t=0,x) = |\alpha\rangle$. Show that it remains a coherent state $\Psi(t,x) = e^{i\theta(t)} |\alpha(t)\rangle$ and determine the time-dependence of the eigenvalue $\alpha(t)$.

(d) Solve the differential equation (1) to find the (normalized) wave function $\Psi(t=0,x) = |\alpha\rangle$.

Using your result from (c), determine $|\Psi(t,x)|^2$ for the time-evolved state. It oscillates in time; Do you recognize the frequency?

3.2 Problem Sheet #3 —— Hand in by Friday 12 December

Problem 3.2.1. For a spin-1 particle, find the matrix representation of the spin-operators S_x, S_y, S_z. Use the basis of eigenvectors of S_z. Verify that S^2 is diagonal in this basis, and that its diagonal entries are $\hbar^2 s(s+1) = 2\hbar^2$ as expected.

Problem 3.2.2. Consider the *three-dimensional* harmonic oscillator $V(\mathbf{r}) = \frac{1}{2}m\omega^2 r^2$. Using separation of variables (in Cartesian coordinates) $\psi(\mathbf{r}) = \psi_x(x)\psi_y(y)\psi_z(z)$ view this as three one-dimensional harmonic oscillators. Determine the allowed energies and their degeneracies.

Problem 3.2.3. The Fourier transform of a three-dimensional wave function $\psi(\mathbf{r})$ is

$$\mathcal{F}\psi(\mathbf{p}) = \widehat{\psi}(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \int_{\mathbb{R}^3} \psi(\mathbf{r}) e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar} \, \mathrm{d}\mathbf{r}.$$

Let ψ be the ground state of Hydrogen. Determine $\widehat{\psi}$.

Hint: Use polar integration letting the polar axis be along the direction of $\mathbf{p}$ and compute the integral over the polar angle first. You may use that $\displaystyle\int_0^\infty s e^{-As} \sin Bs \, \mathrm{d}s = \frac{2AB}{(A^2+B^2)^2}$ for any $A > 0$.

Problem 3.2.4 (Virial theorem)**.**

(a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \mathbf{r} \cdot \mathbf{p} \rangle = 2 \langle T \rangle - \langle \mathbf{r} \cdot \nabla V \rangle \,,$$

where $T = \frac{p^2}{2m}$ is the kinetic energy. Conclude for stationary states that $2\langle T \rangle = \langle \mathbf{r} \cdot \nabla V \rangle$.

(b) Determine $\langle T \rangle$ for the eigenstates of Hydrogen and for the three-dimensional harmonic oscillator. Give your answer in terms of the energies of the respective eigenstates E_n.

Problem 3.2.5. (two-particle problem) Typical interactions between particles depend only on the relative coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, i.e. $V(\mathbf{r}_1, \mathbf{r}_2) = V(\mathbf{r}_1 - \mathbf{r}_2) = V(\mathbf{r})$. In this case, the two-particle problem separates into two independent problems: That in the relative coordinate, and that in the centre-of-mass. To see this, we change coordinates from $\mathbf{r}_1, \mathbf{r}_2$ to $\mathbf{r}$ (the relative coordinate) and $\mathbf{R} = \dfrac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$ (the centre-of-mass coordinate).

(a) Show that

$$\mathbf{r}_1 = \mathbf{R} + \frac{\mu}{m_1}\mathbf{r}, \qquad \mathbf{r}_2 = \mathbf{R} - \frac{\mu}{m_2}\mathbf{r}, \qquad \nabla_1 = \frac{\mu}{m_2}\nabla_{\mathbf{R}} + \nabla_{\mathbf{r}}, \qquad \nabla_2 = \frac{\mu}{m_1}\nabla_{\mathbf{R}} - \nabla_{\mathbf{r}},$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the *reduced mass* of the system.

(b) Show that the time-independent Schrödinger equation in the variables $\mathbf{r}, \mathbf{R}$ becomes

$$-\frac{\hbar^2}{2(m_1+m_2)} \nabla^2_{\mathbf{R}}\psi - \frac{\hbar^2}{2\mu}\nabla^2_{\mathbf{r}}\psi + V(\mathbf{r})\psi = E\psi.$$

Remark: By separating the variables, postulating that $\psi(\mathbf{R}, \mathbf{r}) = \psi_R(\mathbf{R})\psi_r(\mathbf{r})$, we see that ψ_R satisfies the free one-particle Schrödinger equation with the total mass $m_1 + m_2$, and ψ_r satisfies the one-particle Schrödinger equation with potential $V(\mathbf{r})$ and reduced mass μ. In this way, the two-particle problem reduces to a one-particle problem, exactly like the same change of variables does in classical mechanics.